Theory of digital manifolds and its application to medical imaging

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ABSTRACT: Digital manifolds play an important role in computer graphics, 3D image analysis, volume modeling, process visualization, and so forth — in short, in all areas where discrete multidimensional data need to be represented, visualized, processed, or analyzed. The objects in these areas often represent surfaces and volumes of real objects. In this paper we discuss some applications of digital curves and surfaces to medical imaging, implied by theoretical results on digital manifolds.

1 INTRODUCTION

Digital manifolds play an important role in various facets of the modern information society. By becoming a "digital society," the complexity of synthetic digital worlds is increasing. They often represent surfaces and volumes of real objects. This is for example the case in such fields like medicine (e.g., organ and tumor measurements in CT images, beating heart, or lung simulations), bioinformatics (e.g., protein binding simulations), robotics (e.g., motion planning), engineering (e.g., finite elements stress simulations), and security (biometrics). With the rapidly growing variety of synthetic surfaces and volumes, it is becoming critical to develop a relevant theory of digital manifolds and based on it methods for resolving a wide range of problems.

In this paper we review some actual or possible applications in medical imaging implied by some theoretical results on digital manifolds. These applications include visualization of a digitized real object, identification of its topological or geometric properties (such as its tunnels, gaps, skeleton, or boundary), as well as certain metric properties. In Section 2 we refer to some works providing a theoretical basis for the above-mentioned applications that are discussed in Section 3. We conclude with some remarks in Section 4. An extensive bibliography is provided to facilitate interested readers.

2 THEORETICAL FOUNDATIONS

Theory of digital manifolds is a vivid topic of research that is mainly driven by numerous practical applications. In this section we first briefly list and comment some literature sources containing recent developments on the subject. Then we introduce several notions playing an important role in research and related to applications presented in the subsequent sections.

2.1 *Research on Digital Curves and Surfaces*

Before providing a brief overview of results on digital curves and surfaces, we recall a few basic notions. Two 3-cells (voxels) c_1 and c_2 are called α-*adjacent* iff their intersection $c_1 \cap c_2$ contains an α-cell, where $\alpha \in \{0, 1, 2\}$. Alternatively, two grid points $p_1, p_2 \in \mathbb{Z}^3$ are called 6-*adjacent* iff $0 < d_e(p_1, p_2) \leq 1$, 18-*adjacent* iff $0 < d_e(p_1, p_2) \leq \sqrt{2}$, and 26-*adjacent* iff $0 < d_e(p_1, p_2) \leq \sqrt{3}$, where d_e is the Euclidean distance.

Digital surfaces have been studied frequently over the years. For example, (Kim 1983) defines digital surfaces in $\mathbb{Z}^3$ based on adjacencies of 3-cells. A mathematical framework (based on a notion of "moves") for defining and processing digital manifolds is proposed in (Chen & Zhang 1993).

For obtaining α-surfaces by digitization of surfaces in $\mathbb{R}^3$, see (Cohen-Or et al. 1996). It is proved in (Malgouyres 1997) that there is no local characterization of a 26-connected subset S of $\mathbb{Z}^3$ such that its complement $\overline{S}$ consists of two 6-components and every voxel of S is adjacent to both of these components. (Malgouyres 1997) defines a class of 18-connected surfaces in $\mathbb{Z}^3$, proves a surface separation theorem for those surfaces, and studies their relationship to the surfaces defined in (Morgenthaler & Rosenfeld 1981). (Bertrand & Malgouyres 1999) introduces a class of "strong" surfaces and proves that both the 26-connected surfaces of (Morgenthaler & Rosenfeld 1981) and the 18-connected surfaces of (Malgouyres 1997) are strong. For further studies on 6-surfaces, see (Chen et al. 1999). Digital surfaces in the context of arithmetic geometry are studied in (Brimkov et al. 2002). For various other topics related to digital manifolds we also refer to (Chen 2004; Chen 2005).

In a recent paper (Brimkov & Klette 2004) two of the authors provided the first definition of digital manifolds of involving the notion of dimension in discrete spaces (Mylopoulos & Pavlidis 1971). Accordingly, a digital curve is one dimensional while a digital surface is $(n-1)$-dimensional set of voxels, where n is the dimension of the considered discrete space. The definition allows classification of all digital manifolds with respect to the type of their "gaps." The concepts of tunnels and gaps and their relevance to certain practical problems is discussed next.

2.2 *Tunnels, Gaps, and Skeletons*

A *gap* is an important notion in discrete geometry and topology. Usually, gaps are defined through separability as follows: Let a digital object M be m-separating but not $(m-1)$-separating in a digital object D. Then M is said to have k*-gaps* for any $k < m$. A digital object without m-gaps is called m*-gapfree*. See Figure 1.

Homology groups in topology define *tunnels*, and 2-gaps are sometimes also discussed as being tunnels. Information about the number of gaps or tunnels has been a subject of interest in various disciplines, such as digital topology (Fourey & Malgouyres 2002; Ma & Wan 2000; Nakamura 2006; Srihari 1981), image analysis (Kong & Rosenfeld 1989; Lohmann 1988; Saha & Chaudhuri 1996), graph theory (White 1972), and computational modeling of 3D forms (Desburg et al. 2005). Gaps or tunnels are related to important topological concepts such as Euler characteristic and Betti numbers. See (Klette & Rosenfeld 2004) for more details.

For various applications it is useful to obtain the *skeleton* of a digital set. Skeletons represent the basic topological features of the considered object while being easier to study. They are obtained by thinning algorithms. For more details refer, e.g., to (Klette, G. 2006; Klette, G. & Pan 2004; Klette, G. & Pan 2005; Kong 2004; Palagyi & Kuba 2003; Palagyi & Kuba 1998; Palagyi et al. 2001).

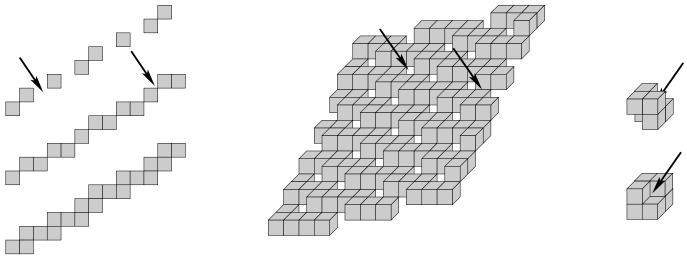

Figure 1: *Left:* From top to bottom: portions of arithmetic lines defined by $0 \leq 3x - 5y < 3$, $0 \leq 3x - 5y < 5$ (naive line), and $0 \leq 3x - 5y < 8$ (standard line). The first one has 1-gaps (and, therefore, also 0-gaps; a 1-gap is pointed out by an arrow), the second one has 0-gaps (one of them pointed out by an arrow) but no 1-gaps, and the third one is gap-free. *Middle:* Portion of an arithmetic plane defined by $0 \leq 2x + 5y + 9z < 7$. It has 2-gaps (and, therefore, also 1- and 0-gaps). A 2-gap and a 1-gap are pointed out by arrows. *Right:* Configuration of voxels (in two different orientations) that features a 0-gap (pointed out by an arrow).

3 APPLICATIONS TO VIZUALIZATION, PROCESSING, AND STRUCTURAL ANALYSIS OF DIGITIZED OBJECTS

In this section we briefly discuss possible applications of digital manifolds, mainly in the area of medical imaging.

3.1 *Finding and Counting Gaps*

Knowledge about gaps is important for ray tracing or understanding of the topology of digitized 3D sets. See (Kaufman 1987; Kaufman 1993; Kaufman 1987a; Kaufman & Shimony 1986). Assume, for example, that an unknown closed continuous surface Γ has been digitized, e.g., by a tomography scanner. Let M be the resulting digital set of voxels. If now the border $\partial(M)$ of M is determined in a way to constitute a digital surface satisfying the proposed definitions, one will have information about the type of possible gaps in that surface. The requirement for gapfreeness of $\partial(M)$ is important when a discrete model of a surface is traced through digital rays (e.g., for visualization or illumination purposes), since the penetration of a ray through the surface causes a false hole in it. Knowledge about the type of gaps of $\partial(M)$ may predetermine the usage of an appropriate type of digital rays for tracing the border in order to avoid wrong conclusions about the topology of the original continuous 3D set having the frontier Γ. Then, for the purposes of surface reconstruction, one will be able to faithfully model the geometry of the original 3D set. This is of importance for 3D imaging, e.g., in medicine.

Information about gaps is also important for ensuring correctness of representation for simulation purposes. For example, a small hole in a heart surface created by imperfections of the synthetic representation, while possibly insignificant (or simply unnoticeable) for visualization, renders the synthetic surface useless for blood flow simulation. Further, finite element simulations may yield incorrect results if surfaces have singularities. Therefore, it is of primary importance to have sound mathematical methods that can assure correctness of key topological, geometric, and metric properties of synthetic surfaces and volumes.

In (Brimkov et al. 2006) the notion of gap was generalized to higher dimensions and the following formula for the number of $(n-2)$-dimensional gaps in a digital object S has been obtained. Let S_k be the set of k-cells of S and $s_i = |S_i|$, $0 \leq k \leq n$. Then

$$g_{n-2} = -2n(n-1)s_n + 2(n-1)s_{n-1} - s_{n-2} + b,$$

where b is the number of $2^2 1^{n-2}$-blocks of S (see (Brimkov et al. 2006) for denotations, definitions, and other details). In particular, the above formula counts the number of 0-gaps and 1-gaps in digital 2D/3D digital objects. A computer program (based on simple linear time algorithm) has been designed to compute the number of 0- and 1-gaps as well as other object parameters. The program also allows to visualize the digital picture S and interactively rotate it along the Ox-, Oy-, and Oz- axes so that the object can be seen from different viewpoints.

3.2 *Number of Tunnels*

Several works address the more difficult and equally important problem of computing the number of tunnels in a digital object. An algorithm from (Saha & Chaudhuri 1996) computes the number of tunnels in a $3 \times 3 \times 3$ neighborhood of a point but not for the whole region. Several other works (Basu 2005; Basu et al. 2005; CHomP & CAPD; De Silva; Kaczynski et al. 2004; Peltier et al. 2005) provide algorithms for the problem, however, with no estimation of the computational complexity.

Using a graph-theoretical approach, in (Li & Klette 2006) the authors present a computationally efficient algorithm with a guaranteed polynomial worst case running time. There is evidence that the same approach could provide an algorithm to compute homology for digitized sets in arbitrary dimension.

3.3 *Visualization, Skeletonization, and Measurements*

Some theoretical developments related to digital manifolds are particularly relevant to the analysis of curve-like structures in biomedical images. An ongoing research project (Klette, G. 2006) at the University of Auckland aims at analyzing confocal microscope images of human brain tissue (which contain cells called astrocytes, see Figure 2, left). These images have been taken layer by layer and constitute a volume defined on a 3D regular orthogonal grid. The curve-like structures have been obtained by applying a thinning algorithm (see Figure 2, right). (Klette, G. 2006) proposes a classification of voxels in 3D skeletons of binarized volumes for subsequent structural analysis and length measurements of digital arcs. For the former, a specific graph is associated with the skeleton (see Figure 3). The nodes of the graph, called *junctions*, exhibit certain interesting properties. However, within the proposed model they are considered as singletons that constitute the set of graph vertices. For the purposes of length measurements, the digital curves are segmented into subsequent maximum-length digital straight-line segments, and the total length of those is used to evaluate the length of the curves. For more details we refer to (Klette, G. 2006).

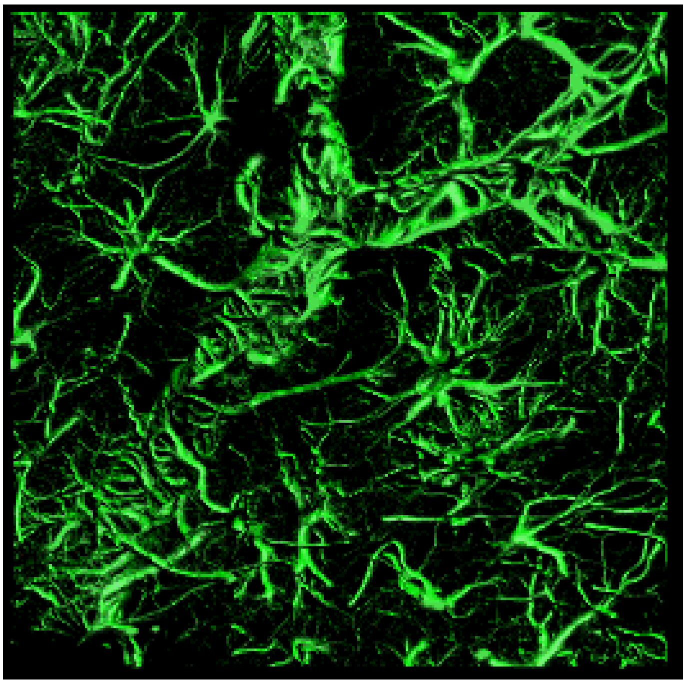

Figure 2: Example of an input data set composed of 42 slices of 256×256 density images generated by confocal microscopy from a sample of human brain tissue.

Note that the arcs of the skeleton form one-dimensional digital curves and as a whole the skeleton is a digital curve satisfying recent definitions from (Brimkov & Klette 2004). These properties support the segmentation process through a number of available efficient algorithms and, in turn, the curve length measurements. Note that curve-like structures appear

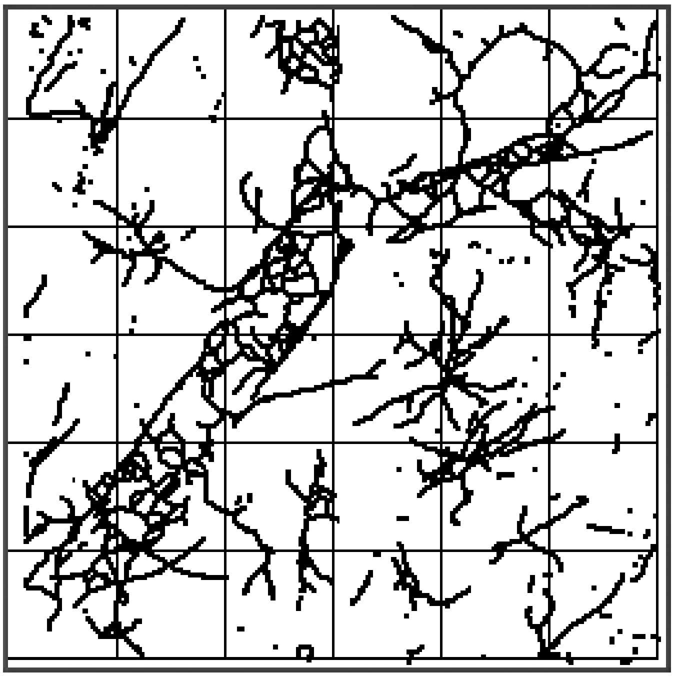

Figure 3: *Right:* A skeleton of the binarized volume shown on the left.

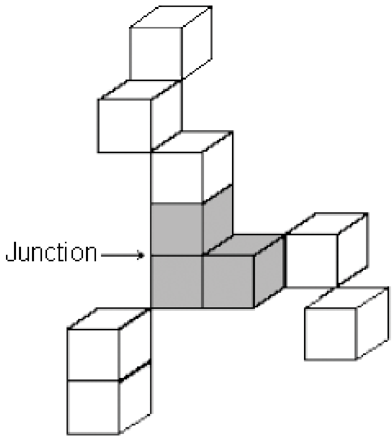

Figure 4: Grey voxels constitute a junction.

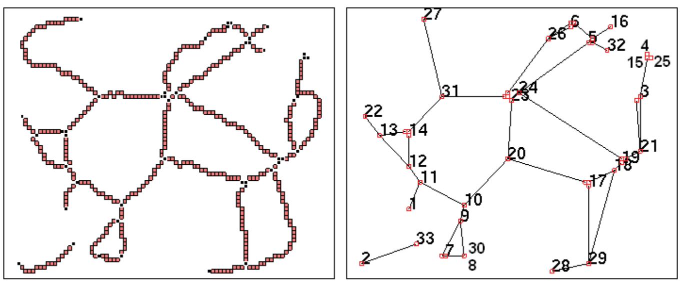

Figure 5: *Left:* Portion of the skeleton from Figure 2 (junctions are shown as small black squares). *Right:* A graph associated with the skeleton from the left. Nodes are labeled by positive integers).

also in other biomedical images, for example in 3D scans of blood vessels or in 3D ultrasound images.

3.4 *Determination of Object Boundary*

Another possible application of the theory of digital manifolds is seen in designing new algorithms for determining the border of a digital object. Because of its importance, this problem has attracted considerable attention (see, e.g., (Daragon 2005; Kovalevsky 1989; Latecki 1988) and the bibliographies in those).

Our hypothesis is that one would benefit from an algorithm that constructs the border as a digital surface as defined in (Brimkov & Klette 2004). As already discussed earlier, the reason for this is the knowledge about the gaps in the surface.

If a digital object has been obtained by digitizing a set with a “regular” shape (e.g., featuring convexity), then, in practice, the border voxels indeed constitute a digital surface satisfying those definitions. Moreover, for data compression purposes the obtained digital surface can be “linearized” by partitioning it into polygonal portions of digital planes. The fact that any digital plane is a digital surface explains why in practice the requirement for two-dimensionality supports the minimization of the number of digital plane patches. For more details we refer to (Klette & Sun 2001).

In some cases however it is possible that the border voxels of a digital set do not constitute a digital surface. This usually happens when the digital object has a very complex and irregular structure. An illustration of such a complexity is provided in Figures 6, 7, and 8. They present digitized images of a human brain tissues, studied within the previously mentioned astrocyte project. In such cases, one possibility is to algorithmically “repair” the set of border voxels in order to make it two-dimensional. Some theoretical results from (Brimkov & Klette 2004) suggest that such a digitization always exists. Repairing digital objects in order to achieve desired properties has been already used by some researchers (e.g., (Siguara et al. 2005; Latecki 1988)).

4 CONCLUDING REMARKS

The purpose of this paper was to introduce the reader to ongoing research on properties of digital manifolds and related applications to medical imaging. Mathematically sound foundations may guarantee high quality rendering of objects, faultless simulations (e.g., of organ functions), and computational efficiency of the image analysis and processing. In order to achieve optimal effect, theoretical research should go in parallel with applied work. Close collaboration between specialist with diverse expertise will become increasingly important.

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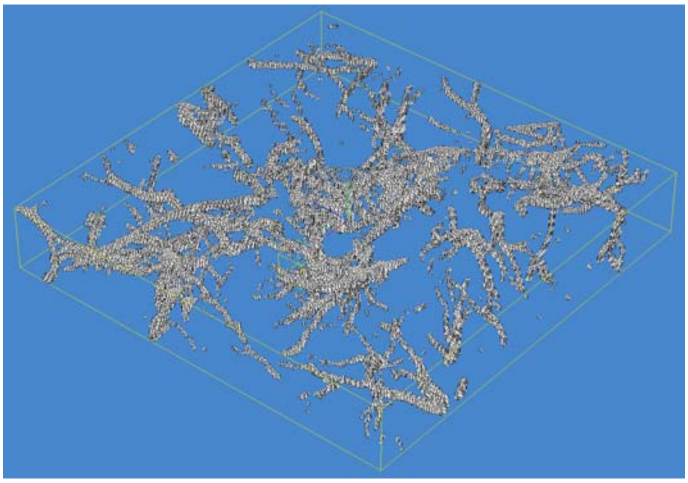

Figure 6: Large view of a sample of human brain tissue, studied within the astrocyte project. The data have been obtained by confocal microscopy and visualized in voxel view mode.

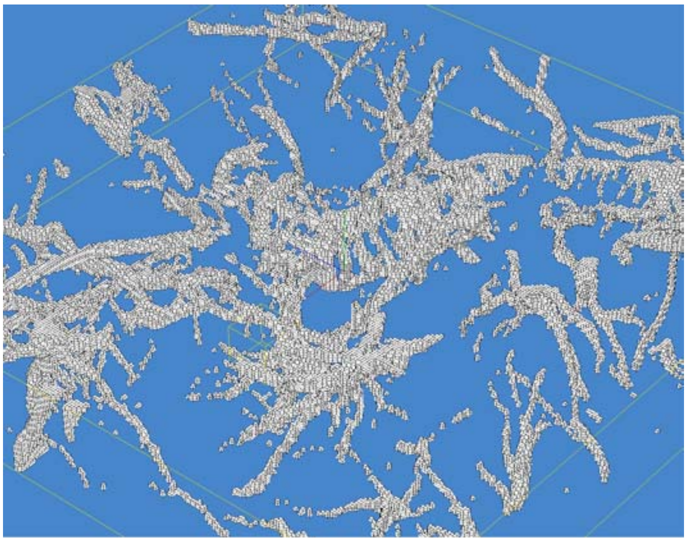

Figure 7: Enlarged view of a detail of the volume in Figure 6.

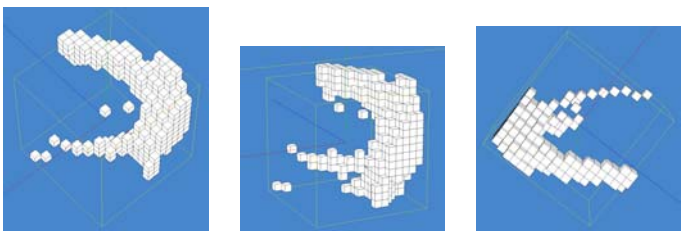

Figure 8: Further enlargements of subvolumes of the digital image of Figure 2.

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