DIGITAL GEOMETRY

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REFERENCES

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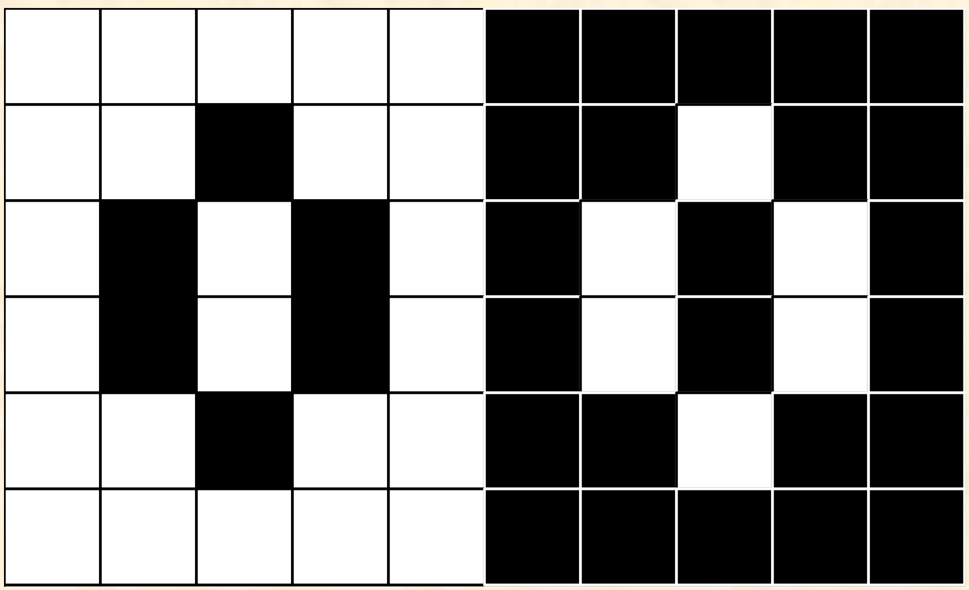
R. Klette and A. Rosenfeld
DIGITAL GEOMETRY: GEOMETRIC METHODS FOR DIGITAL IMAGE ANALYSIS,
Morgan Kaufmann, San Francisco, 2003, in preparation.

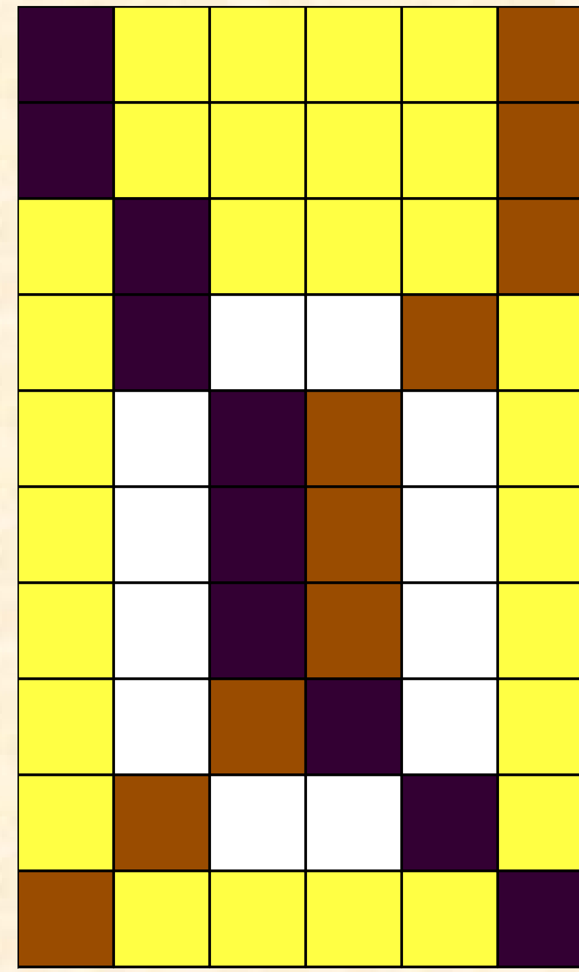
DIGITAL GEOMETRY

is the study of geometric properties of subsets of digital images.

binary images



multi-level images



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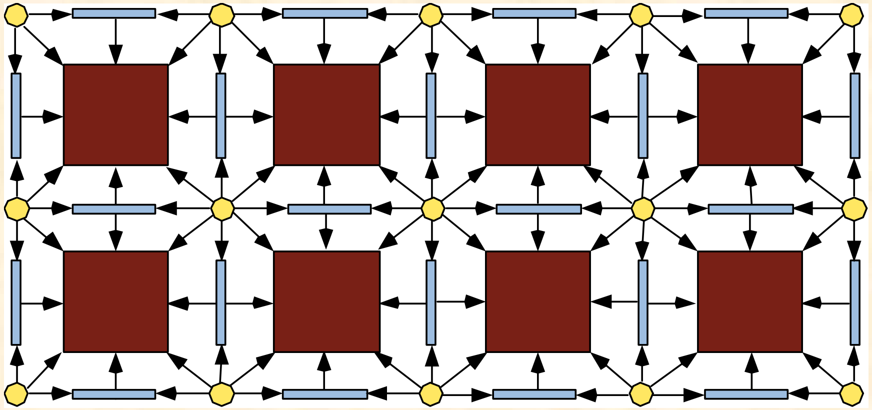
OUTLINE

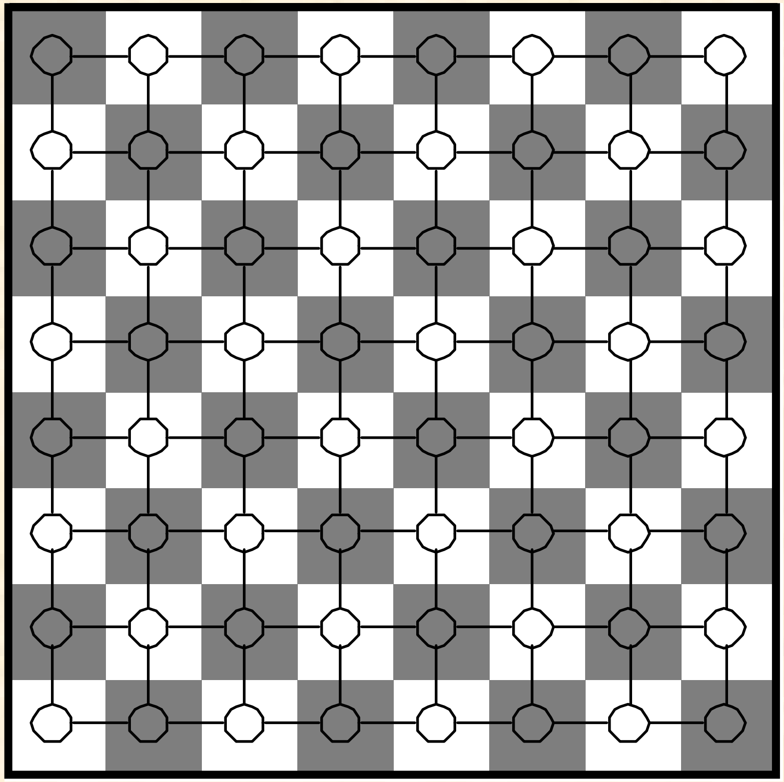
1. Digital spaces and images
2. Digital geometry as discrete geometry
3. Digital geometry as approximate Euclidean geometry
4. Digital geometry vs. computational geometry
5. Generalizations

1. DIGITAL SPACES AND IMAGES

- **Lattice points and adjacency relations**
  Rosenfeld/Pfaltz 1966
  Rosenfeld 1970

- **Cell complexes**
  Listing 1861, Steinitz 1908 ...
  Herman 1981, Kovalevsky 1989





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Basic models/theories for digital spaces and images:

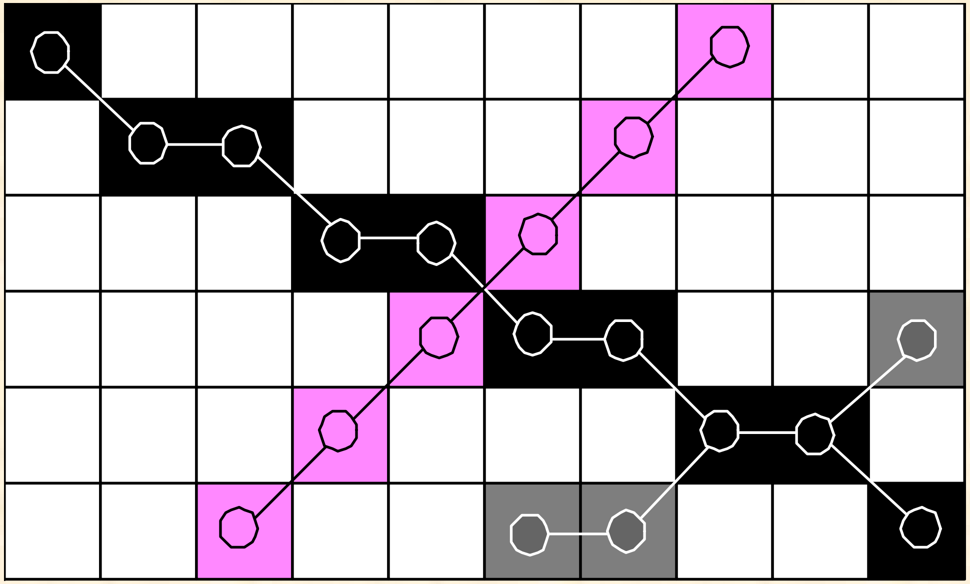
- 4-, 6-, 8-neighborhoods in 2D, good pairs for 3D
- adjacency graph models, neighborhood structures
- poset topology (Khalimsky-Kovalevsky plane)
- inter-pixel boundaries, half-integer grid
- oriented adjacency graphs (Voss et al.), combinatorial maps
- theory of n-dimensional cell complexes
- combinatorial topology
- ...

SOME ODDITIES compared to Euclidean spaces:

"points", "lines", ... have rather different properties than they do in Euclidean spaces:

points (cells) can be neighbors

lines are sequences of isolated points (cells) and can intersect in segments



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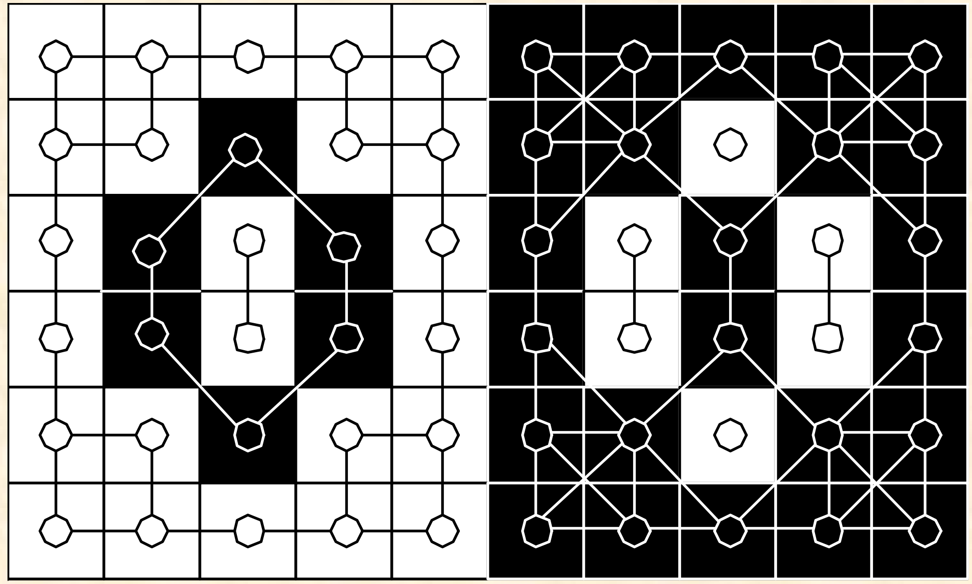
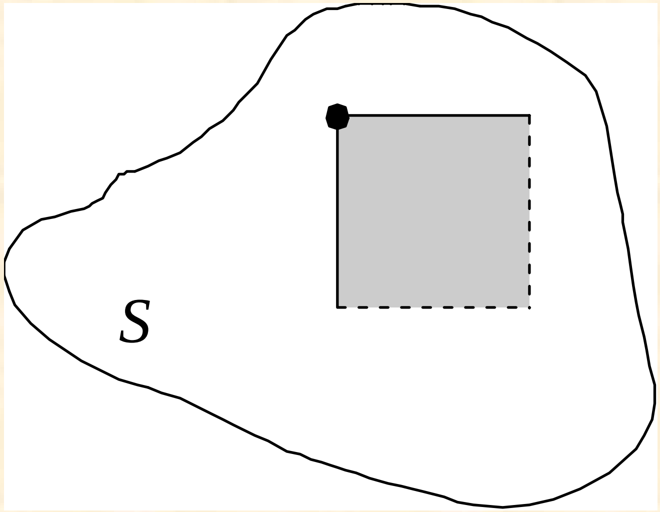


4-adjacency **8-adjacency**

SOME BASICS OF DIGITAL TOPOLOGY

- Define the DIGITIZATION D = <S> of a subset S of the Euclidean plane as the union of the half-open square cells that intersect S.



- Define a digital set D to be 8-CONNECTED if any two pixels of D are joined by a sequence of pixels of D such that successive pixels of the sequence are 8-adjacent.

- THEOREM: D = <S>, where S is connected in the Euclidean topology, iff D is 8-connected.

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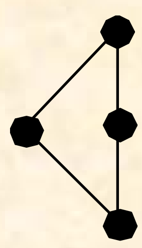
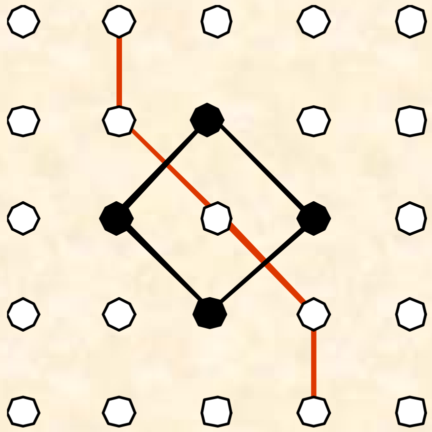
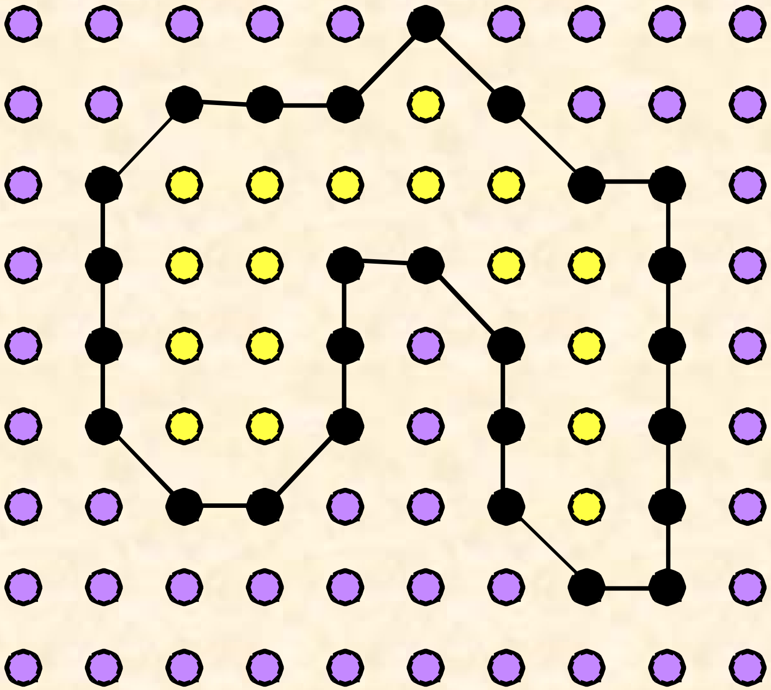
THE NEED FOR TWO TYPES OF CONNECTEDNESS

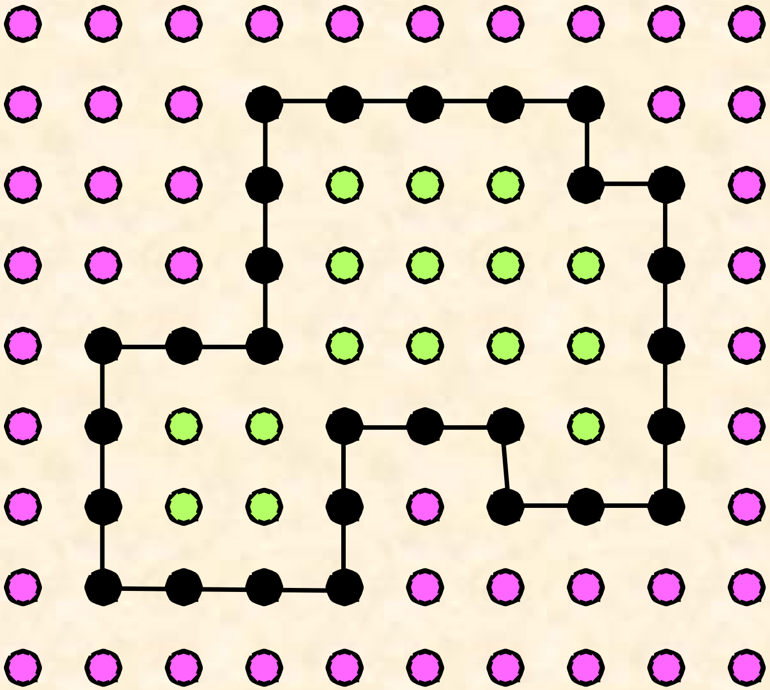
- Unpleasant fact:
  If we use only this definition of 8-connectedness,
  a closed path doesn't separate the plane into an inside
  and an outside - e.g.





- Solution:
  also use 4-connectedness by not allowing diagonal neighbors;
  use opposite types of connectedness for a set and its complement





CONSEQUENCES

- If a component of D and a component of its complement are adjacent, one of them surrounds the other.

- The complement of a closed path has exactly two components (the “Jordan curve theorem”).

- The EULER CHARACTERISTIC of a set D (the number of components of D minus the number of components of its complement) is locally computable.

- Connectedness properties are preserved by adding/deleting “simple” pixels to/from a set.

(A pixel is SIMPLE if its neighborhood intersects just one component of the set and one component of its complement.)

EXAMPLES OF GEOMETRIC CONCEPTS, PROPERTIES, AND RELATIONS

- Adjacency
- Neighborhoods
- Borders and interiors
- Arcs and curves
- Pathwise connectedness
- Pathwise distance
- Area, perimeter, extent, diameter, etc.
- Elongatedness (and “thinning”)
- Arc length and curvature
- Intrinsic distance
- Convexity and straightness

2. Digital geometry as discrete geometry

ISSUES

- Definitions of geometric properties

- Complexity of computing the properties

- Local computability

- Characterizing image operations that preserve the properties

- Characterizing digital objects that could be digitizations of
  Euclidean objects that have given properties

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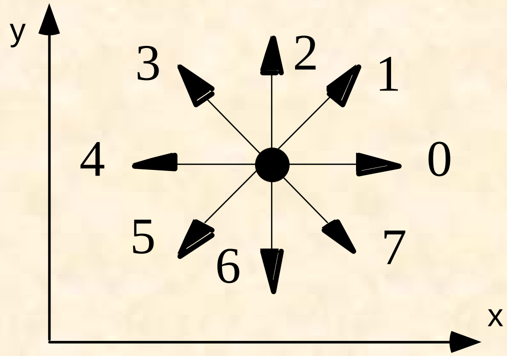
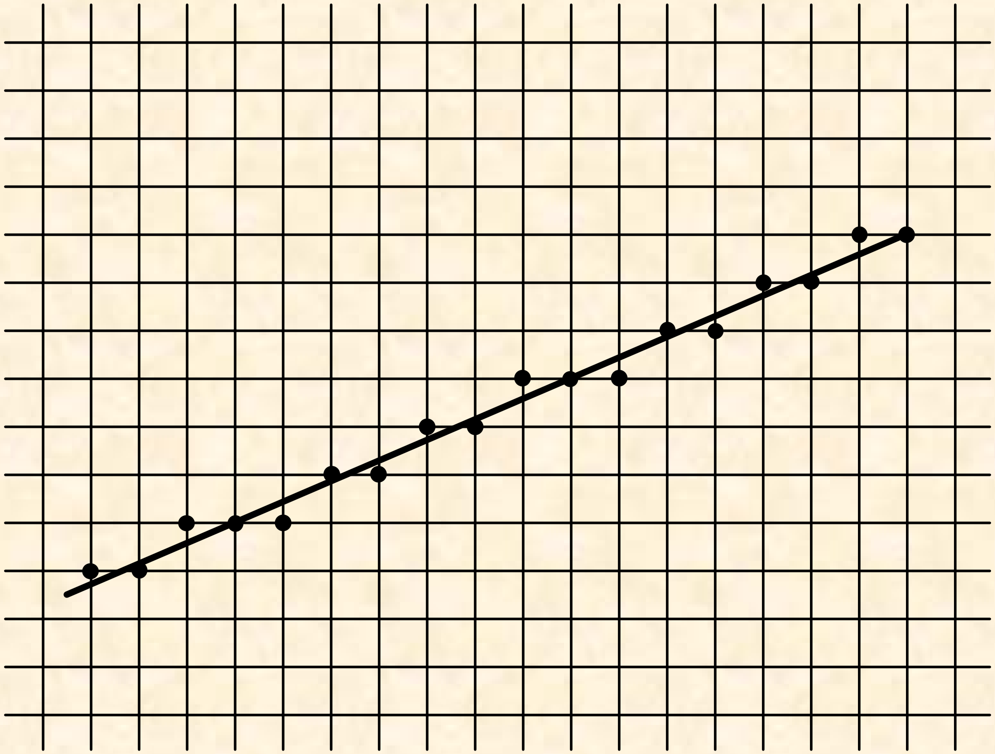
Grid-intersection digitization

Freeman 1961





digital straight segment (DSS) =

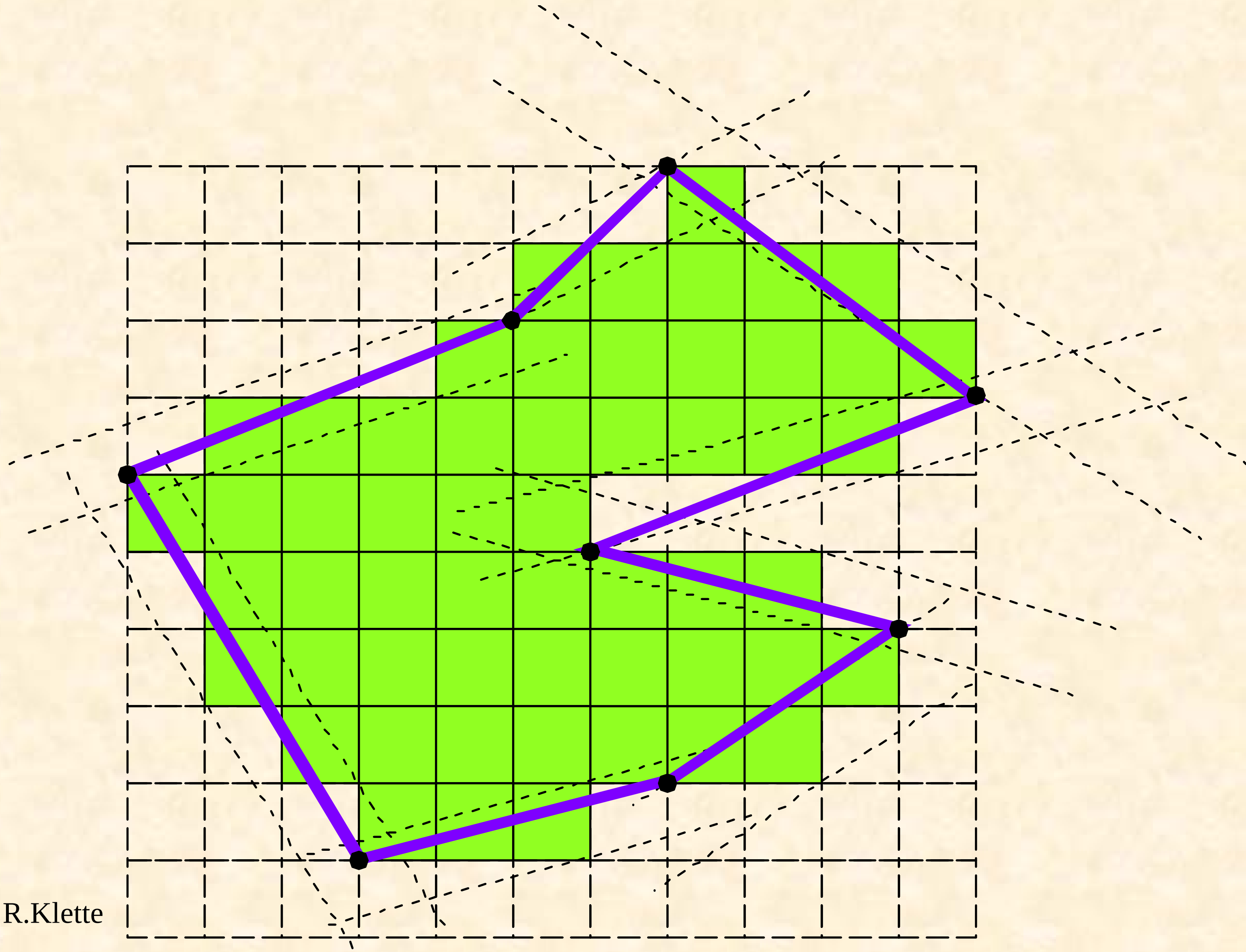
8-curve resulting from a Euclidean straight line segment

Rosenfeld 1974: **(i)** a DSS is an irreducible 8-arc.

(ii) A finite irreducible 8-arc is a DSS iff it satisfies the chord property.

Basic routine in image analysis:

segmentation into maximum-length DSSs



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A few contributions towards O(n) on-line DSS recognition:

1976 **J. Rothstein & C. Weiman**

first layer only of off-line linguistic DSS algorithm

1981 **A. Hübler, R. Klette & K. Voss**

linear off-line DSS algorithm: linguistic approach

1982 **L.D. Wu**

linear off-line DSS algorithm: linguistic approach (minor flaw)

1982 **C.E. Kim**

brief sketch of linear off-line CSS algorithm
(based on Sklansky's convex hull algorithm)

1982 **E. Creutzburg, A. Hübler, & V. Wedler**

two linear on-line DSS algorithms:
(a) linguistic approach and (b) geometric approach

1983 **S. Shlien**

linear off-line DSS algorithm: linguistic approach

1985 **T.A. Anderson & C.E. Kim**

sketch of linear off-line DSS algorithm

1988 **E. Creutzburg, A. Hübler, O. Sykora**

linear on-line DSS for specifying a separability problem for monotone polygons

1988 **E. Creutzburg, A. Hübler, O. Sykora**

linear on-line DSS algorithm

1990 **V.A. Kovalevsky**

linear on-line DSS for 4-connected sequences

1991 **A.W.M.Smeulders & L. Dorst**

linear off-line DSS , correcting Wu 1982

1995 **I. Debled-Rennesson & J.-P. Reveilles**

linear on-line DSS , also correcting Wu 1982

and many more .....

3. Digital geometry as approximate Euclidean geometry

If an image can be digitized sufficiently finely, properties of a subset of the “real” image should be adequately approximated by properties of a subset of the digital image.

On the other hand: Digital spaces and images allow studies of geometric properties of subsets, either in the context of graph theory or of combinatorial topology.

The **question** arises how digital (graph-theoretical or combinatorial) concepts correspond to concepts of *digitized Euclidean geometry*.

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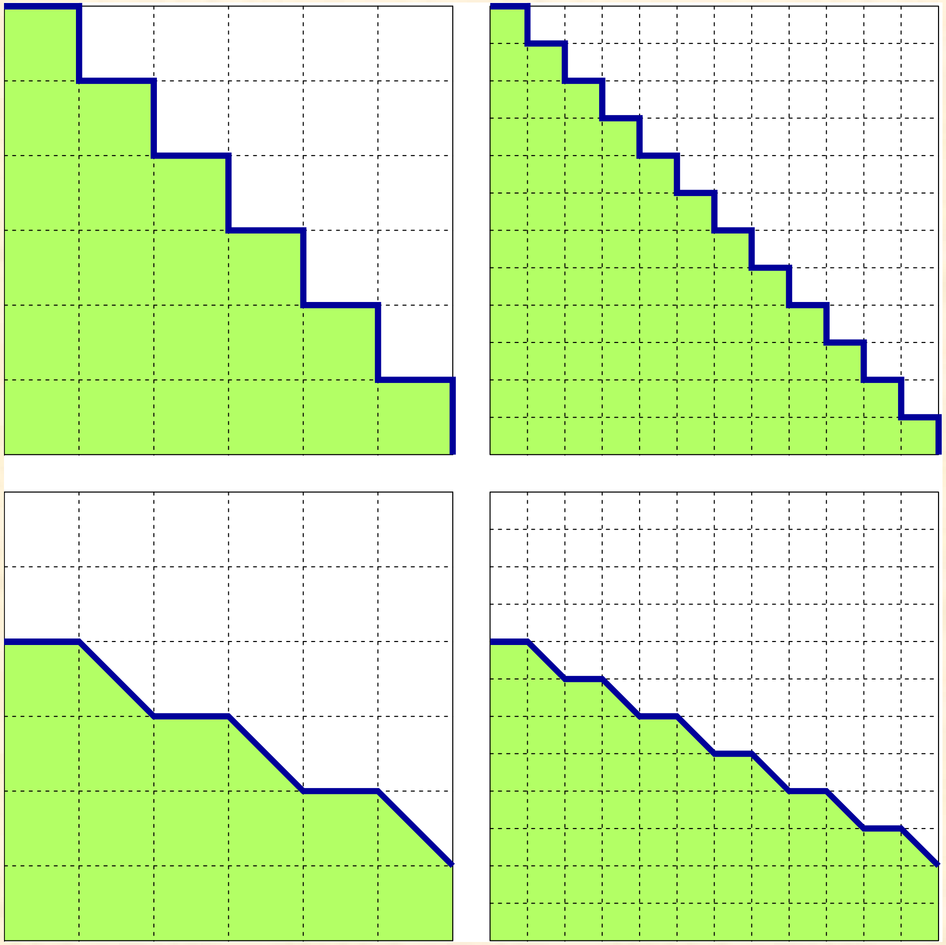
APPROXIMATE EUCLIDEAN GEOMETRY: ISSUES

Multigrid Convergence of Properties:
As the grid becomes finer, do the digital property values (such as length, a moment, etc.) converge to the Euclidean property values? If so, how fast is the convergence with respect to grid resolution?

Multigrid Convergence of Sets:
As the grid becomes finer, do the digitally constructed sets (such as convex hulls, medial axes, etc.) converge to the Euclidean analogs? If so, how fast is the convergence?

EXAMPLE OF NON-CONVERGENCE:

Digital arc length exceeds true arc length, and doesn't approach it in the limit (diagonal/staircase).



Maximal DSS (digital straight segment) approximation is an example of a multigrid-convergent method.

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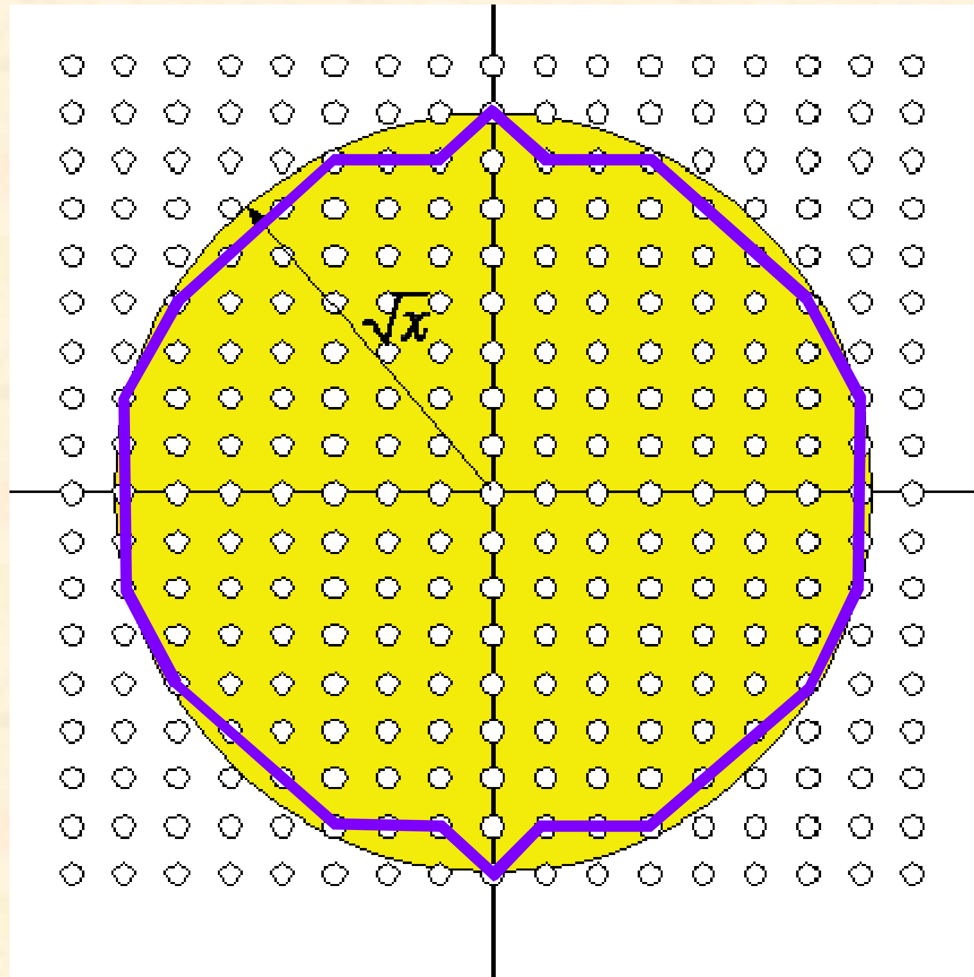
EXAMPLE OF NON-CORRESPONDENCE OF CONCEPTS:

A digitized circle doesn't have the smallest (digital arc length) perimeter of all objects having a given area.



Shorter 8-border possible if digitizing a diamond having the same area

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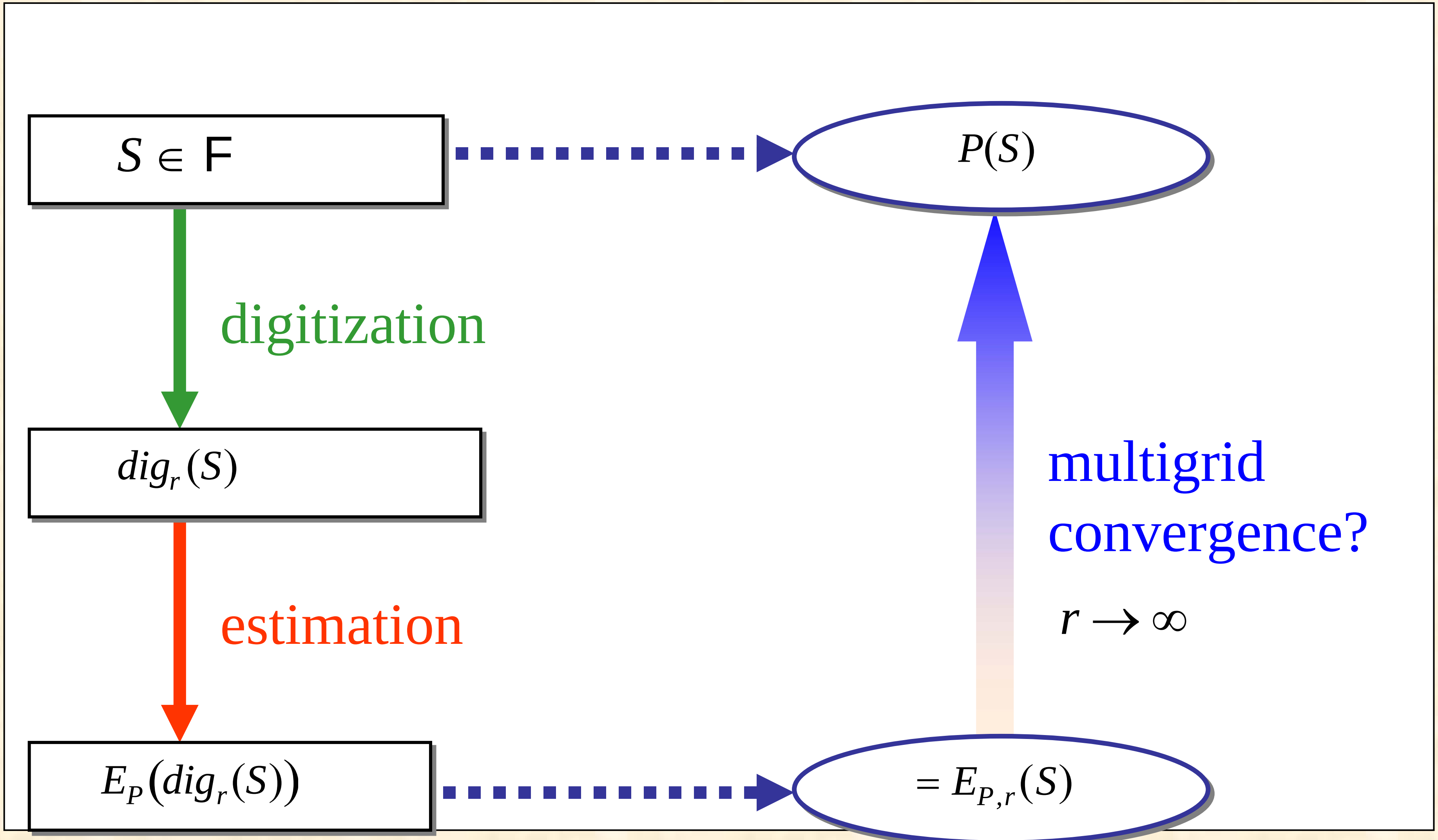
MULTIGRID CONVERGENCE

Serra 1982 (for sets, not properties)

length of grid edge = 1/r



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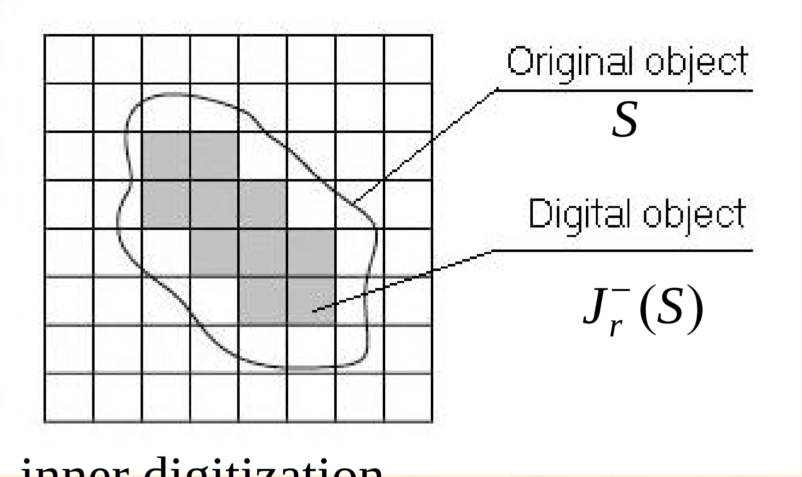
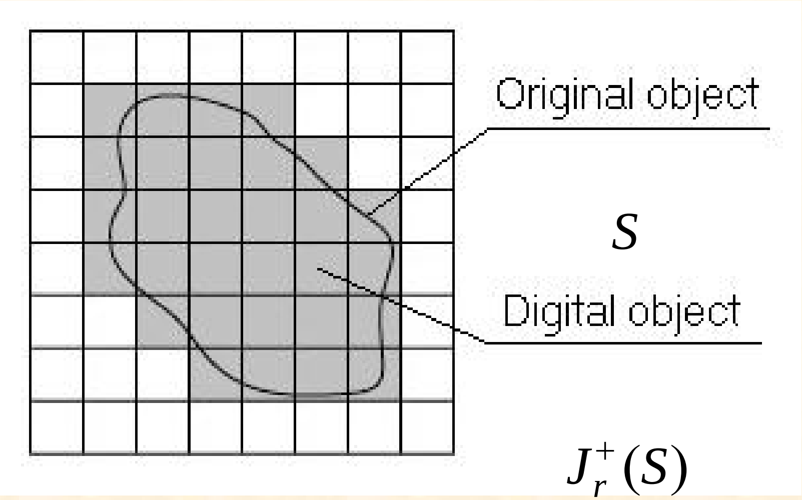
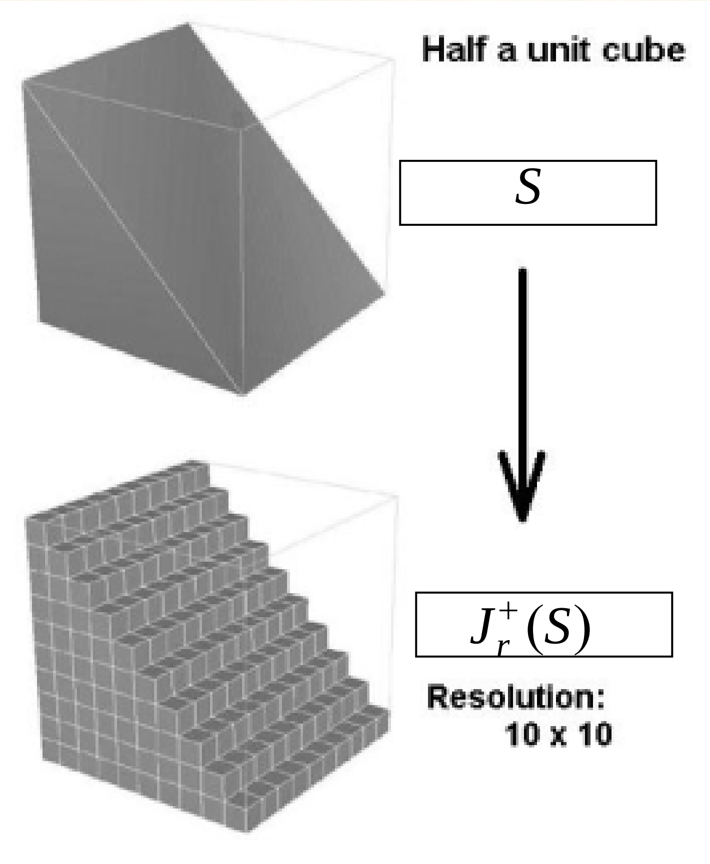
Another digitization model: Jordan digitization in 2D, 3D, ...



inner digitization



outer digitization



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Jordan, Peano 1892: volume estimation in 3D
20th century: generalizations to n-dimensional case

Let S be an n-dimensional set with a Jordan boundary. Then

$$Vol(S) = \lim_{r \to \infty} Vol(I_r(S)) = \lim_{r \to \infty} Vol(O_r(S))$$

in 2D:
known since Gauss (~1820) that this convergence has linear speed

in nD:
studies on speed of convergence for moments (including volume)

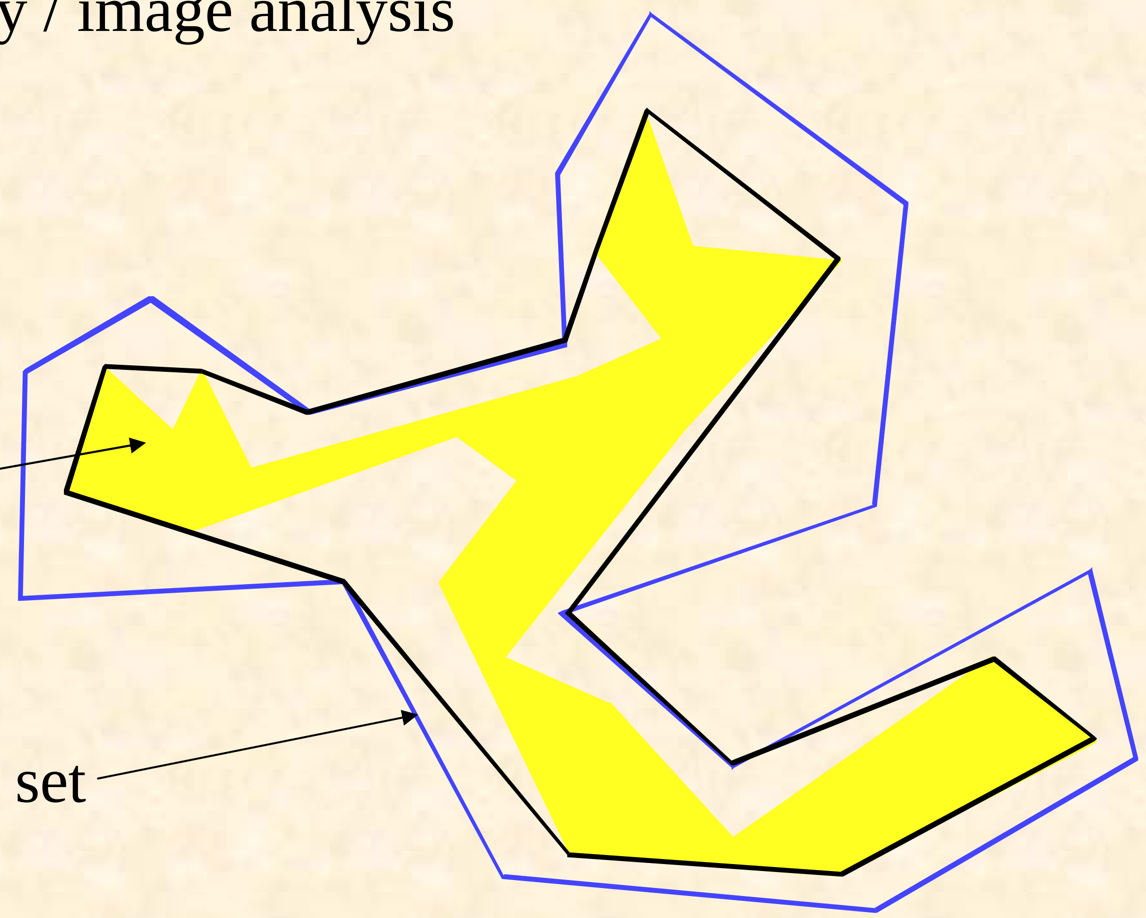
4. Digital geometry vs. computational geometry

Computational geometry deals with finite collections of (N) simple objects (points, lines, circles,...) in Euclidean space, and studies the complexity of computing properties of the collections as N increases.

In digital geometry, the objects don't behave like Euclidean objects (as we have seen).
Also, for practical purposes, digital image size is bounded; reducing the order of complexity of a computation is only of interest if asymptotic constant remains "reasonably small".

Relative convex hull

Sklansky and Kibler 1976: definition of the *relative convex hull* in the context of digital geometry / image analysis



given set

convex hull relative to this set

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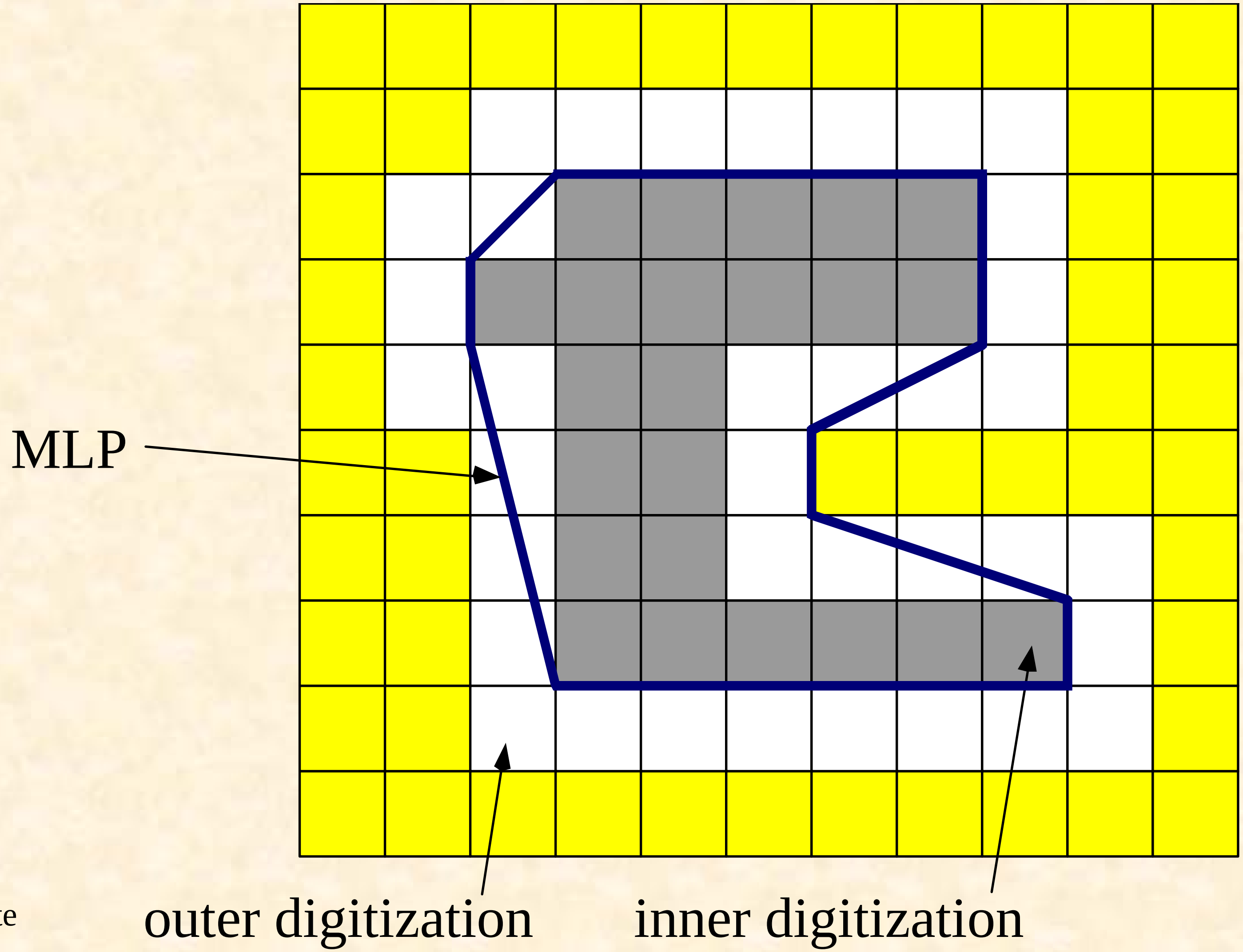
In 2D: relative convex hull = *minimum-length polygon* (MLP) circumscribing the given set, contained in the bounding set



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$A \subseteq Q \subset R^3$ is *Q-convex* iff for all $p, q \in A$

$pq \subseteq Q$ then $pq \subseteq A$

relative convex hull $CH_Q(P)$ of P with respect to Q

= intersection of all Q-convex sets containing P

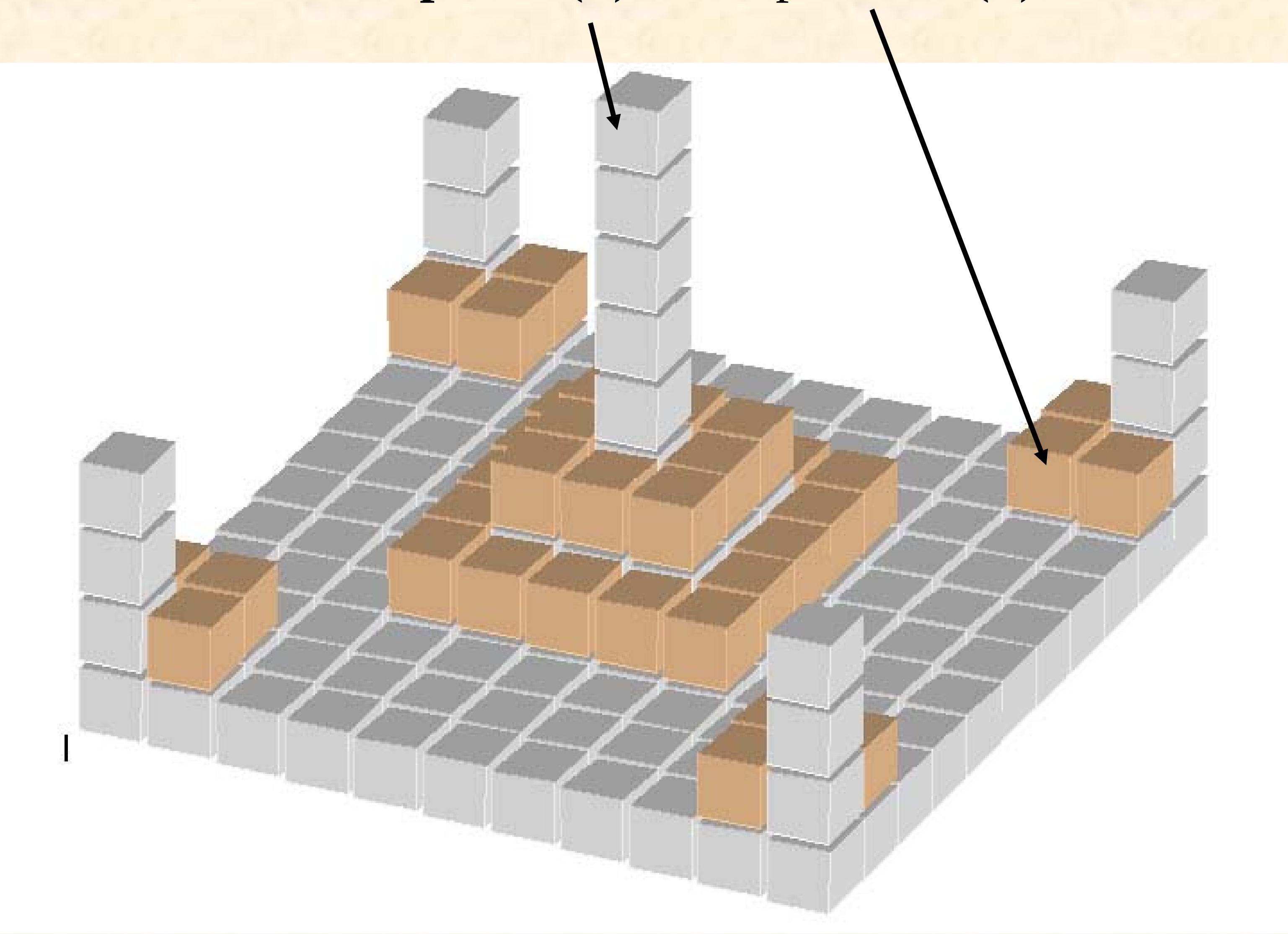
Theorem (Sloboda and Zatko 2001)

S be a compact set in 3D space
bounded by a smooth closed Jordan surface

then

$$\lim_{r \to \infty} Area\left(CH_{J_r^+(S)}\left(J_r^-(S)\right)\right) = Area(S)$$

Example: $J^-(S)$ and part of $J^+(S)$



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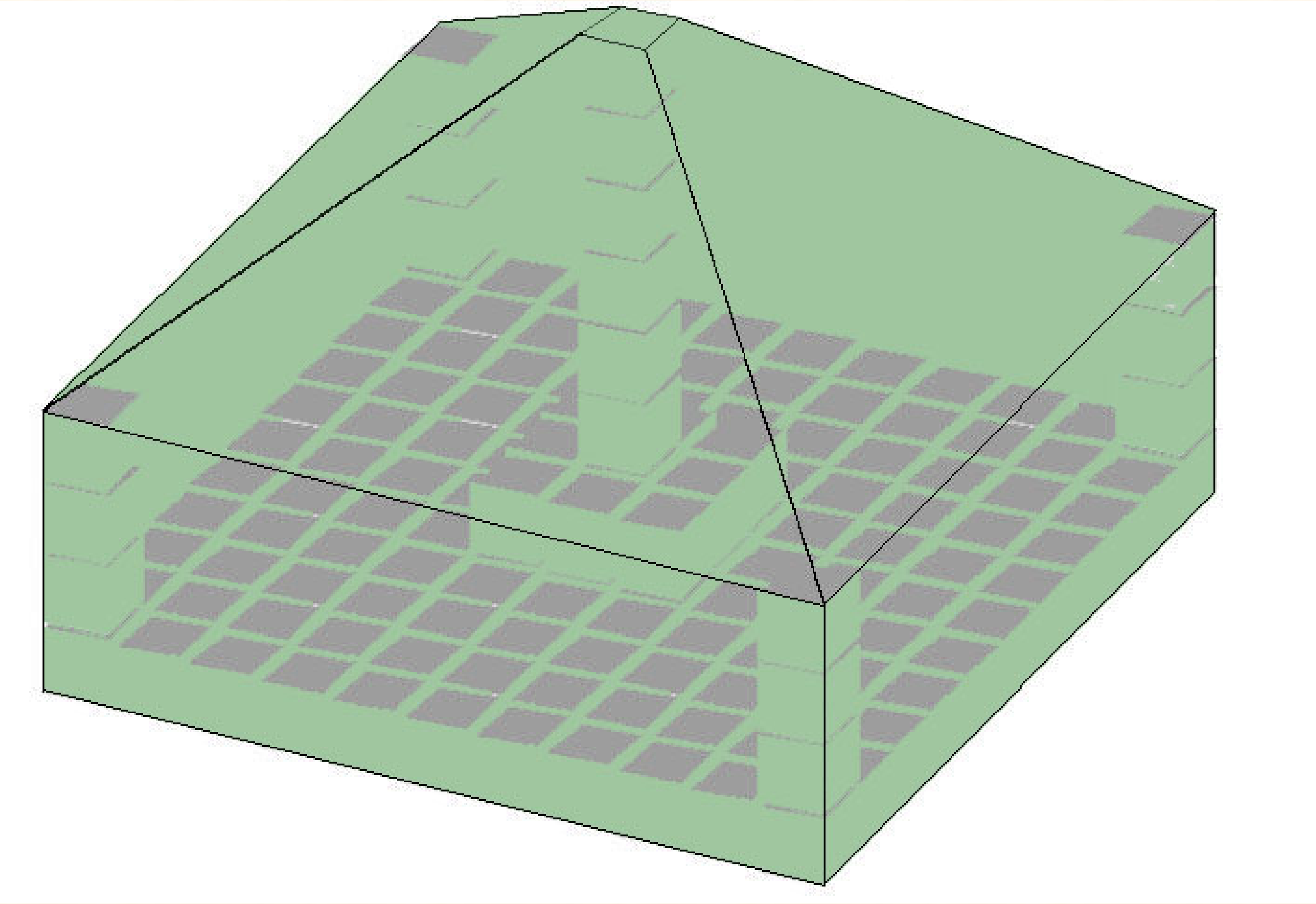
CH(J$^-$(S))



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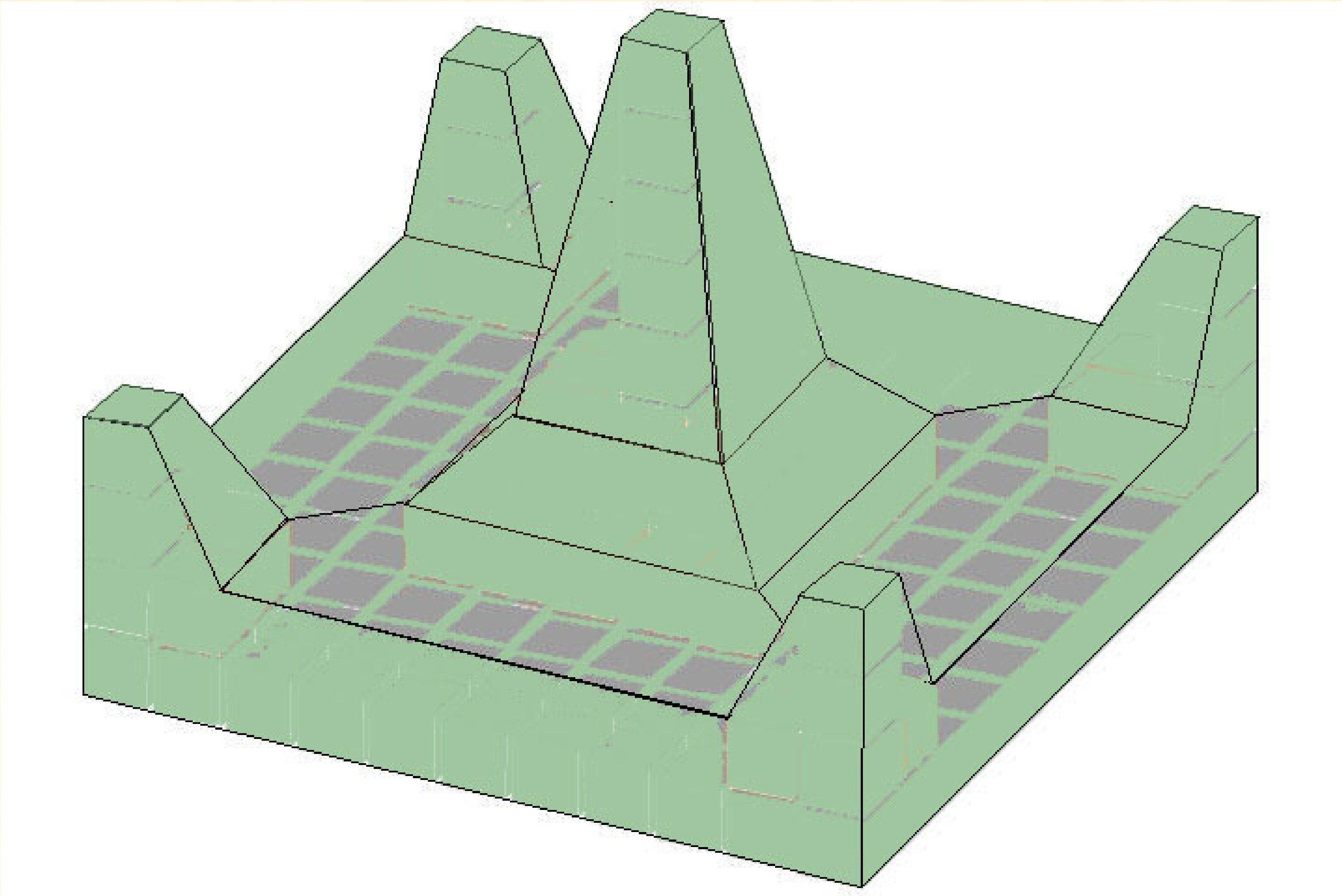
$CH_{J+}(J^{-}(S))$



Open problem: efficient algorithm for relative convex hull in 3D

see list of open problems on: http://www.citr.auckland.ac.nz/dgt

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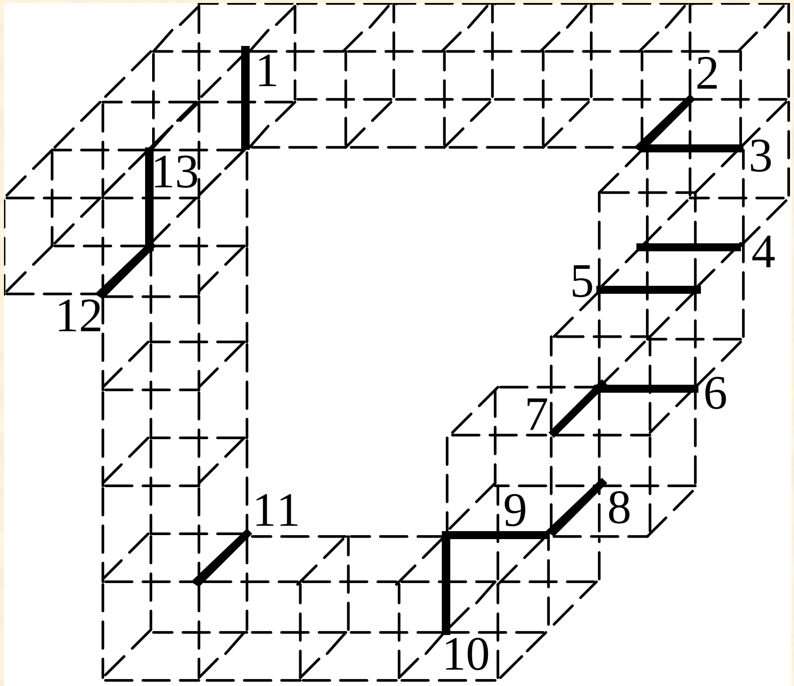
relative convex hull proved to be important in
robotics, CAD, graphics, ...

many algorithmic studies in computational geometry for 2D case

MLP is multigrid convergent length estimator for digitized curves

however:

- no studies on MLPs in computational geometry for 3D (length estimation for 3D digital curves)
- also no studies on relative convex hulls in 3D (surface area estimation)



revival of joint meetings (digital and computational geometry) is recommended

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5. Generalizations

- Higher dimensions
- “Good pairs” in topology
- Surfaces
- Non-standard grids or tessellations
- Abstract discrete spaces
- Fuzzy subsets

web site on digital geometry: citr.auckland.ac.nz/dgt