Digital Geometry
-
The Birth of a New Discipline

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Contents of Talk

- **DIGITAL GEOMETRY**

  THREE CLASSIC PAPERS BY AR & J

  DIGITAL GEOMETRY
  traditional anno 1979 / revised 198

  MULTIGRID CONVERGENCE

- **LENGTH AND SURFACE AREA**

  CURVES in 2D
  two techniques

  SURFACES in 3D
  two techniques

- **A FEW OPEN PROBLEMS**

picture taken in 2000

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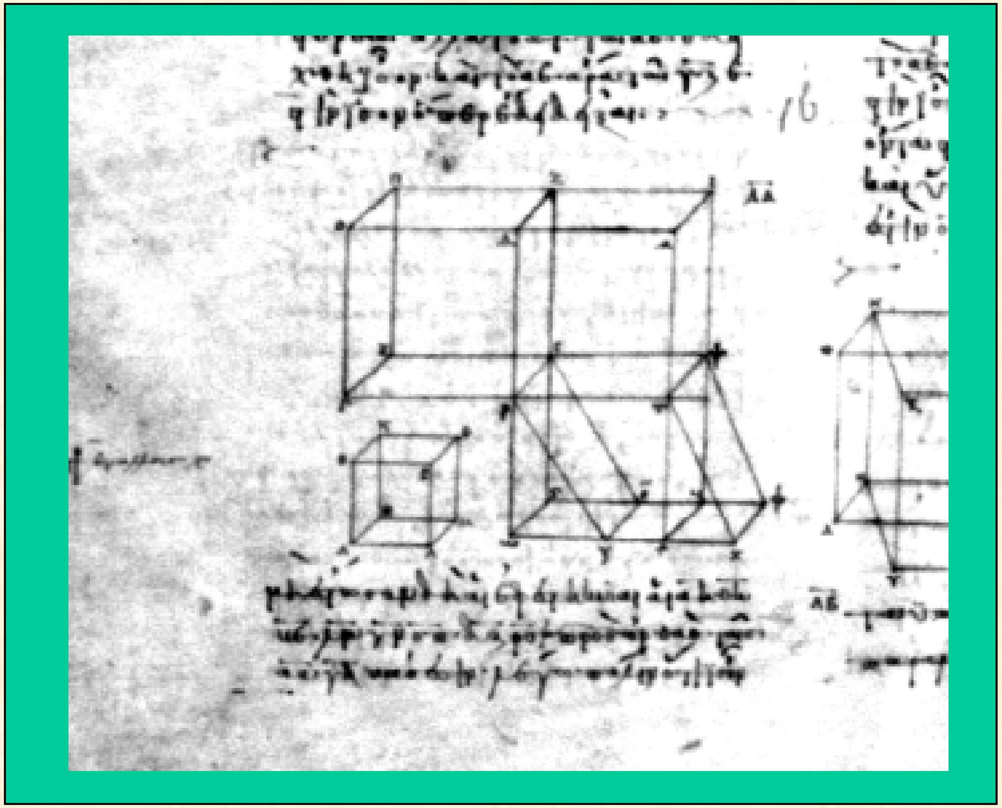


figure in **Euclid**'s ``Elements'', Book XI Propositions 31--33.
Today we could call these `face-connected cubes'.

"It's magic, or geometry, or one of those things."

Terry Pratchett

Euclidean g.
analytical g.
perspective g.
projective g.
descriptive g.
non-Euclidean g.
combinatorial g.
similarity g. of polyhedra
affine g. of polyhedra
projective g. of polyhedra
….

computer g.
computational g.
digital g.
discrete g.
…

1872 Felix Klein: Erlangener Programm

Geometry = a basic manifold **B**
a set of figures $\mathsf{F} \subseteq 2^{\mathbf{B}}$
a group **G** of transformations defined on **B**

Geometric discipline

- a meaningful combination of $\mathbf{B}$, $\mathsf{F} \subseteq 2^{\mathbf{B}}$, $\mathbf{G}$
- significance in science or technology

$\mathbf{B}$ requires a *structure*, such as being a linear, metric or topological space, or just with a system of neighborhoods $U(x) \subseteq \mathbf{B}$

y is proper neighbor of x iff $y \in U(x)$ and $x \neq y$

Digital geometry

- **B** = grid / cells, F = regions, **G** = a group on **B**
- fundamental for computational imaging (worldwide)

Structure on **B**

- grid: neighborhoods (**Rosenfeld/Pfaltz** **1966**)
- cells: bounding relation (**Listing** **1861/1862**)

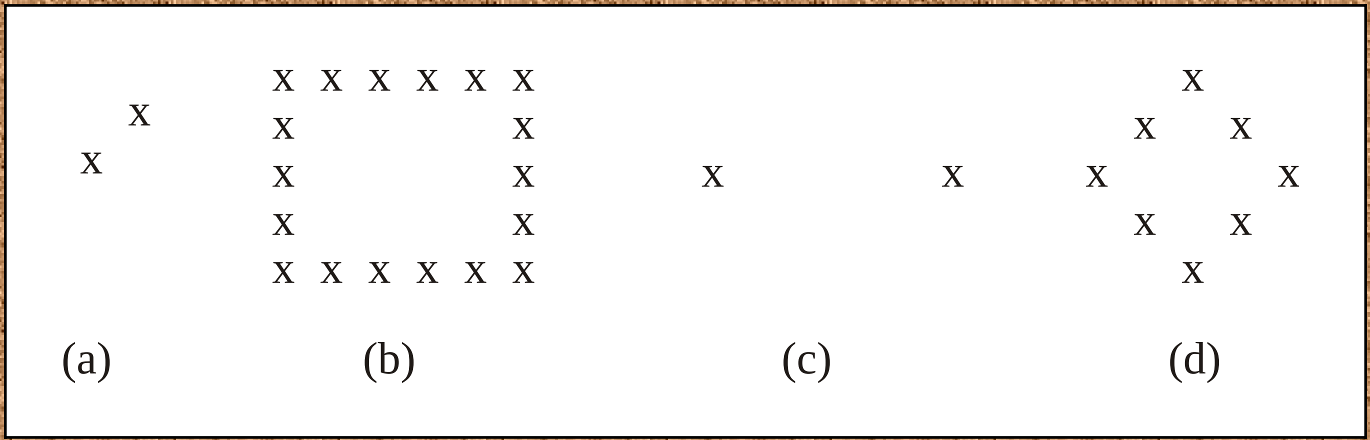
THREE CLASSIC PAPERS BY AR & JLP

on connectedness, distance transforms, and metrics on grids

A. Rosenfeld and J.L. Pfaltz.
Sequential operations in digital picture processing.
J. ACM, 13:471--494, **1966** .

J.L. Pfaltz and A. Rosenfeld.
Computer representation of planar regions by their skeletons.
Comm. ACM, 10:119--122, 125, **1967** .

A. Rosenfeld and J.L. Pfaltz.
Distance functions on digital pictures.
Pattern Recognition, 1:33--61, **1968** .



1966

``the `paradox' of Figure (d) can be expressed as follows: If the `curve' of shaded points is connected (`gapless'), it does not disconnect its interior from its exterior; if it is totally disconnected it *does* disconnect them. **This is of course not a mathematical paradox**, but it is unsatisfying intuitively; nevertheless, connectivity is still a useful concept. It should be noted that if a digitized picture is defined as an array of hexagonal, rather than square, elements, the paradox disappears''

1970/73 **A. Rosenfeld**

Theorem: 4-curves separate 8-holes and
8-curves separate 4-holes

concept used in binary image processing:

use 4-connectivity for objects and 8-connectivity for background

first *major stimulus* for **digital topology**:

ensure duality of separation and connectivity in spaces used in digital geometry (grids, cells)

1966 **A. Rosenfeld and J.L. Pfaltz**

labeling of connected components

computation of distance transforms (based on **Blum** and **Kotelly**)

skeleton = local maxima of 8-radii

second *major stimulus* for **digital topology**:

distance transforms for shape simplification …

shrinking / medial axes / thinning in 2D /
thinning in 3D / thinning in grey level images

"skeleton" today: not the original meaning anymore

TRADITIONAL DIGITAL GEOMETRY

1979 **A. Rosenfeld** (book chapter on digital geometry)

"By *digital geometry* we mean the mathematical study of geometrical properties of digital picture subsets."

SUBJECT LIST

- segmentation of pictures
  (arc, curve, digital Jordan curve theorem)
- simplification of picture subsets
  (simple points)
- measurements of picture subsets
  (area, perimeter, genus, shape factor)
- graph metrics for distances in pictures
  (intrinsic diameter, geodesic, chord length)
- and … (straightness, convexity, homotopy, …)

THREE INTERRELATED AREAS

- *Digital topology*
  open/closed, connected, genus, simple points, skeleton,…
- *Graph theory and combinatorics*
  length/metrics based on neighborhoods, centers in graphs, …
- *Digital geometry*

1987 **A. Rosenfeld and R.A. Melter**

"*Digital geometry* is the study of geometric properties of sets of lattice points produced by digitizing regions or curves in the plane."

SUBJECT LIST now starts in 1987 with

digitization, digital convexity, digital straightness

DIGITAL GEOMETRY =

digitized Euclidean geometry

in regular grid or cell spaces

1961 H. Freeman **1963 J. Bresenham**

digital straight lines - digitization and synthesis

1968 A. Rosenfeld and J.L. Pfaltz

approximation of Euclidean distance by integer metrics

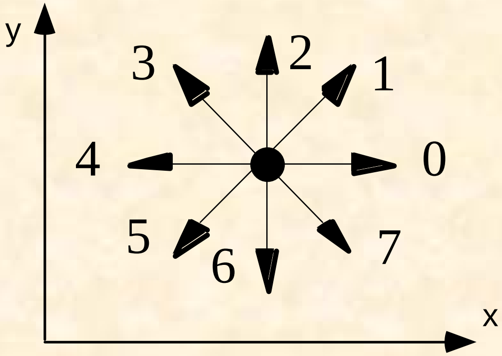
1974 A. Rosenfeld

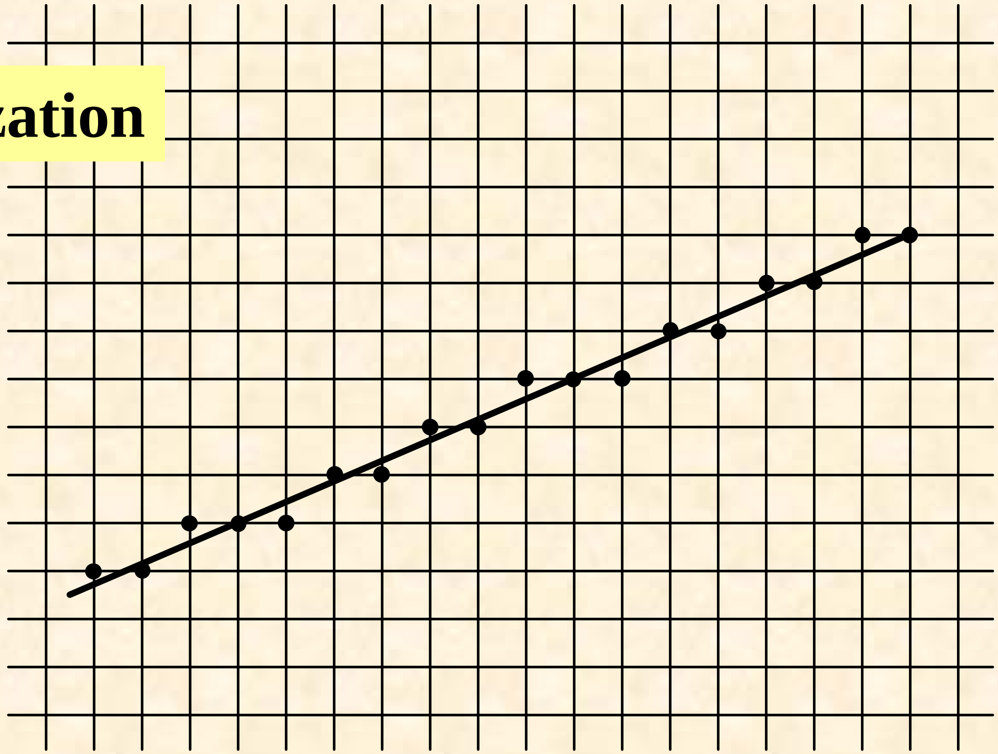
digital straight segments - characterization by chord property

DIGITIZATION MODELS

1961 **H. Freeman**

grid-intersection digitization

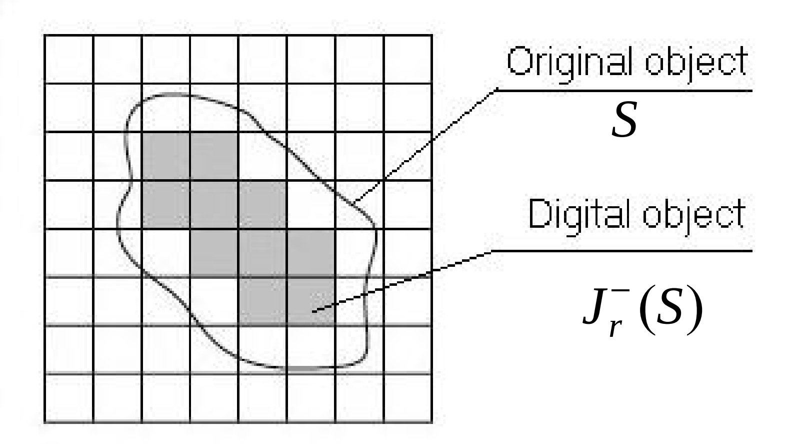




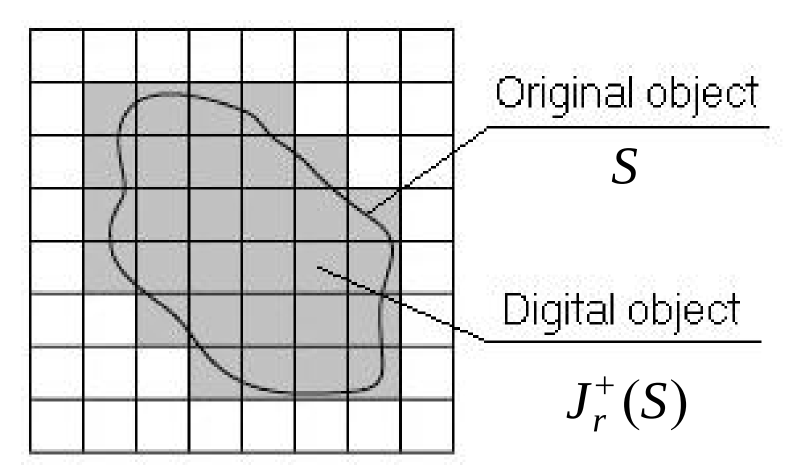
digital straight line = 8-curve resulting from a (real) straight line

(excluding $y = x + i/2$)

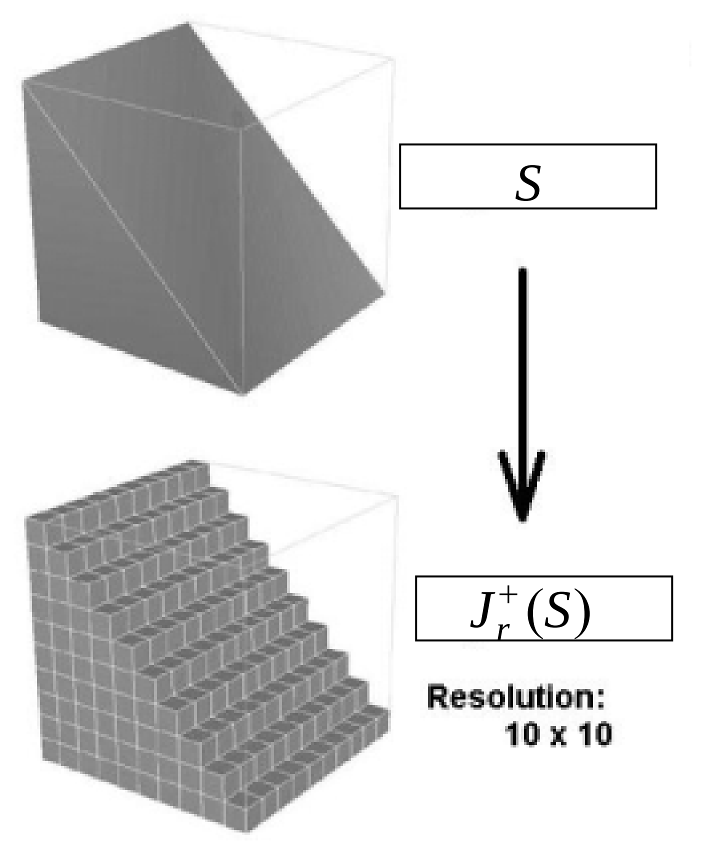
Jordan digitization in 2D, 3D, ...



inner digitization



outer digitization



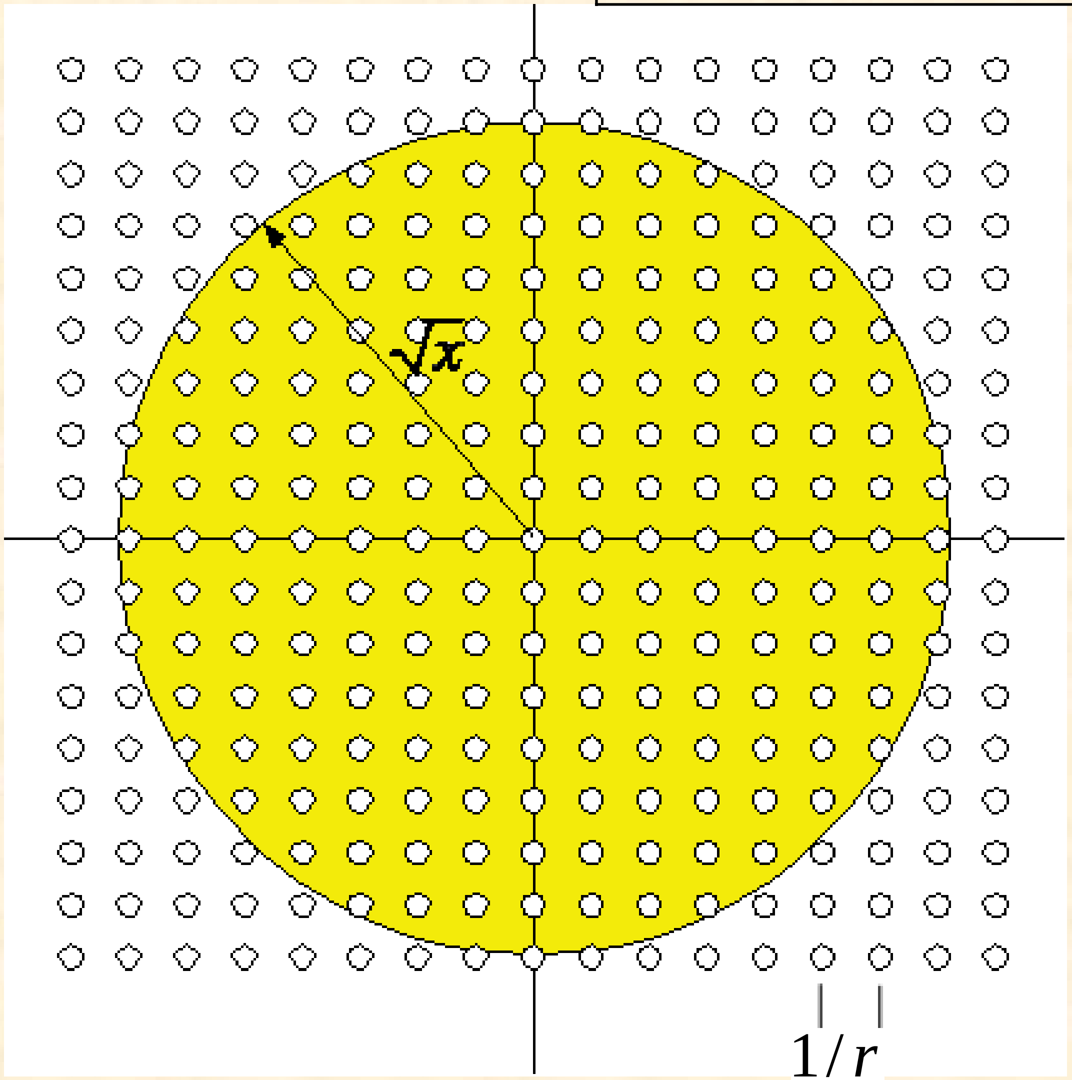
1892...

Jordan, Peano, Minkowski, Scherrer, … - *multigrid contents estimation*

~1820 **C.F. Gauss**

Gauss digitization in 2D

$$G_r(S) = \{(i/r, j/r) : (i/r, j/r) \in S \wedge i, j \text{ integers}\}$$



$$A(S) = A(G(S)) + O(\sqrt{x})$$

$G(\mathbf{S})$ = Gauss digitization

$$f(r) = |A(S) - A(G_r(S))| = O(r^{-1})$$

r = grid resolution

linear convergence $c \cdot \frac{1}{r}$

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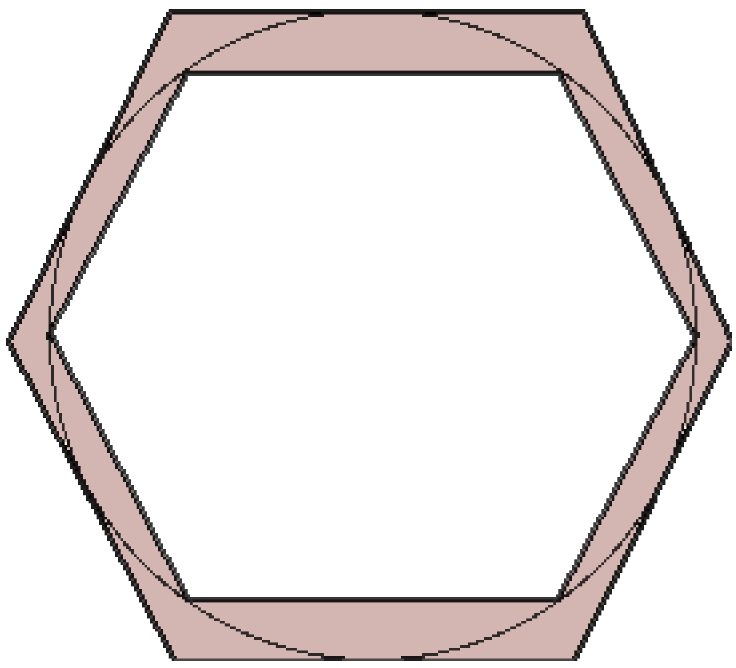
287 BC - 212 BC **Archimedes** 3rd century **Liu Hui** - *estimation of* π

$$n_m = 3 \times 2^m = 3, 6, 12, 24, 48, 96$$

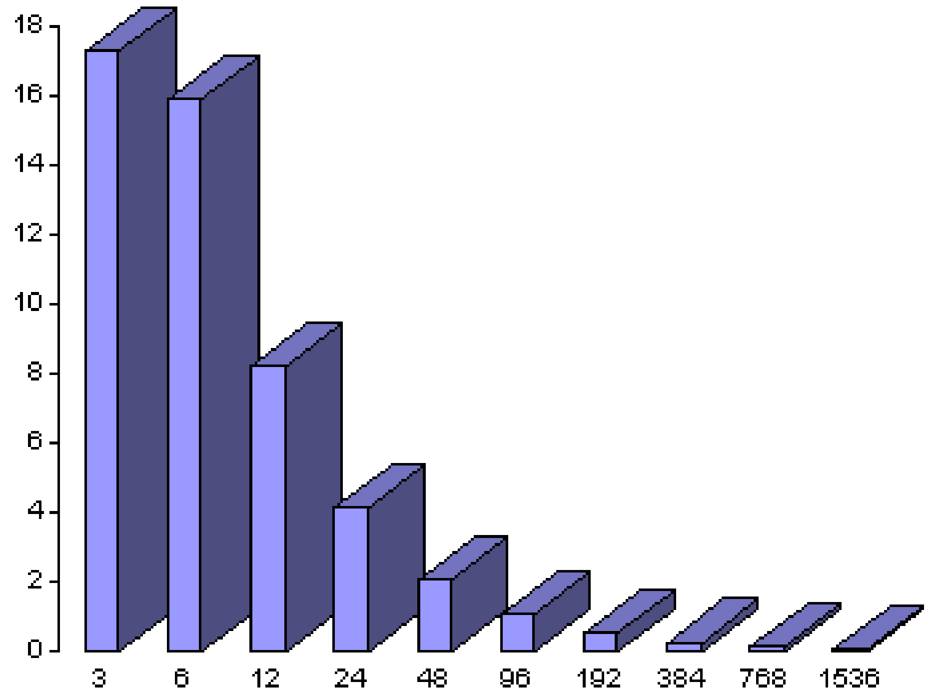
$$p_{m+1} = \rho \sqrt{2\rho^2 - \rho \sqrt{4\rho^2 - p_n^2}}$$

linear convergence $c \cdot \frac{1}{n}$

$$f(n) = |p_n - 2\pi\rho| \approx 2\pi\rho / n$$



inner and outer hexagon



percentage of errors for inner *n*-gon

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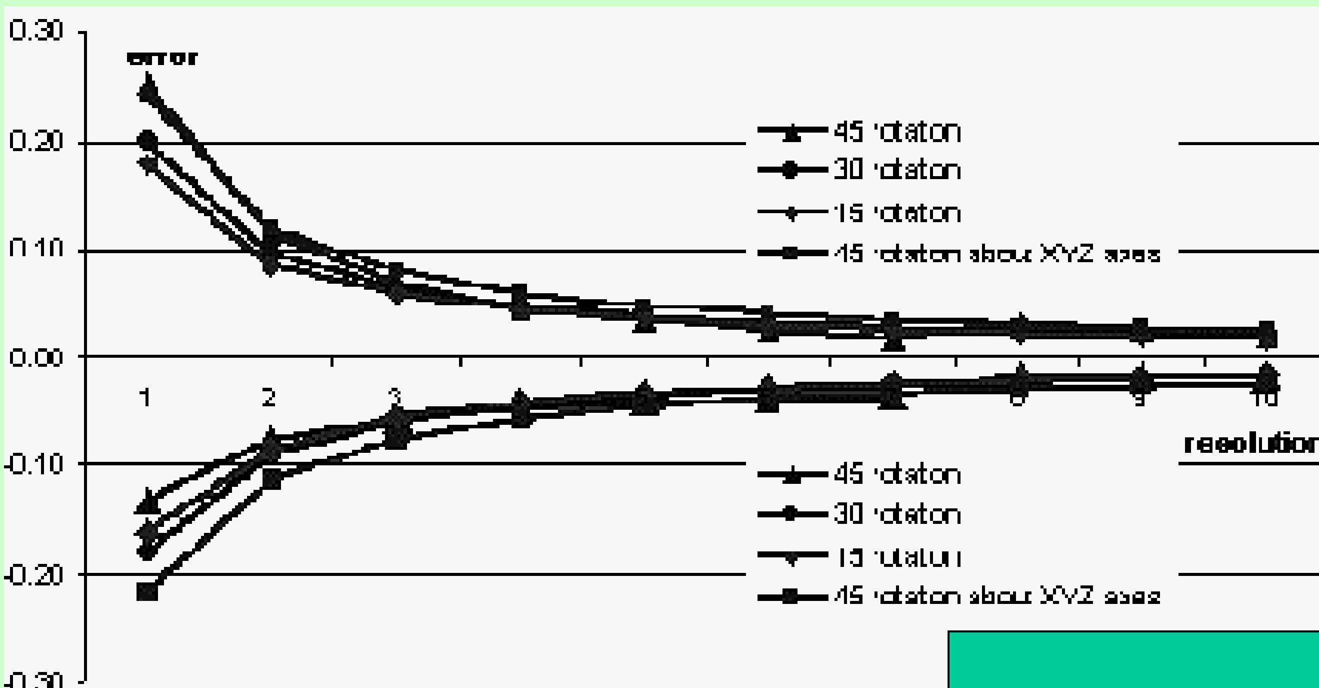
1892 **Jordan, Peano:** *volume estimation*

Let S be a regular solid. Then

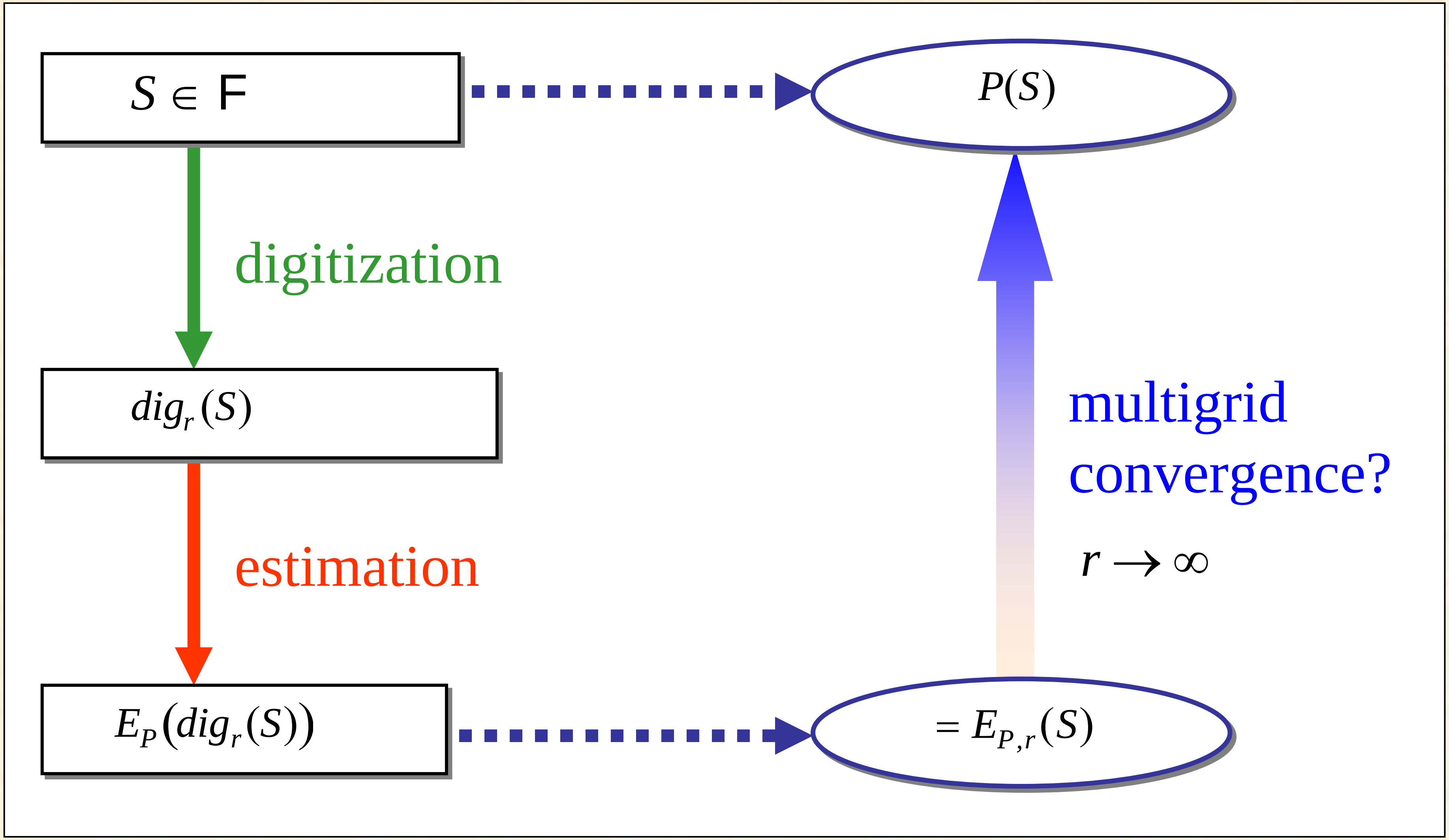
$$V(S) = \lim_{r\to\infty} I_r(S) = \lim_{r\to\infty} O_r(S)$$

Experiment: cube in different rotational positions and $r \to \infty$



linear convergence $c \cdot \frac{1}{r}$

MULTIGRID CONVERGENCE



1982 **J. Serra** (for sets)

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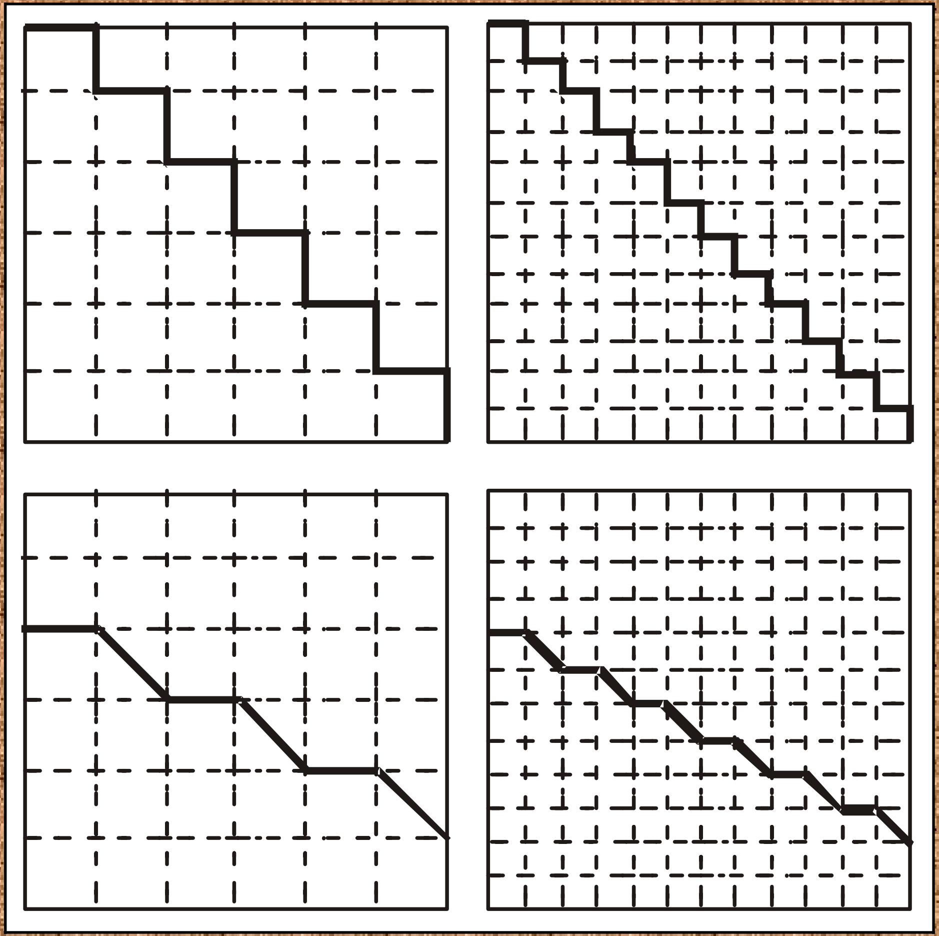
LENGTH OF A CURVE IN 2D

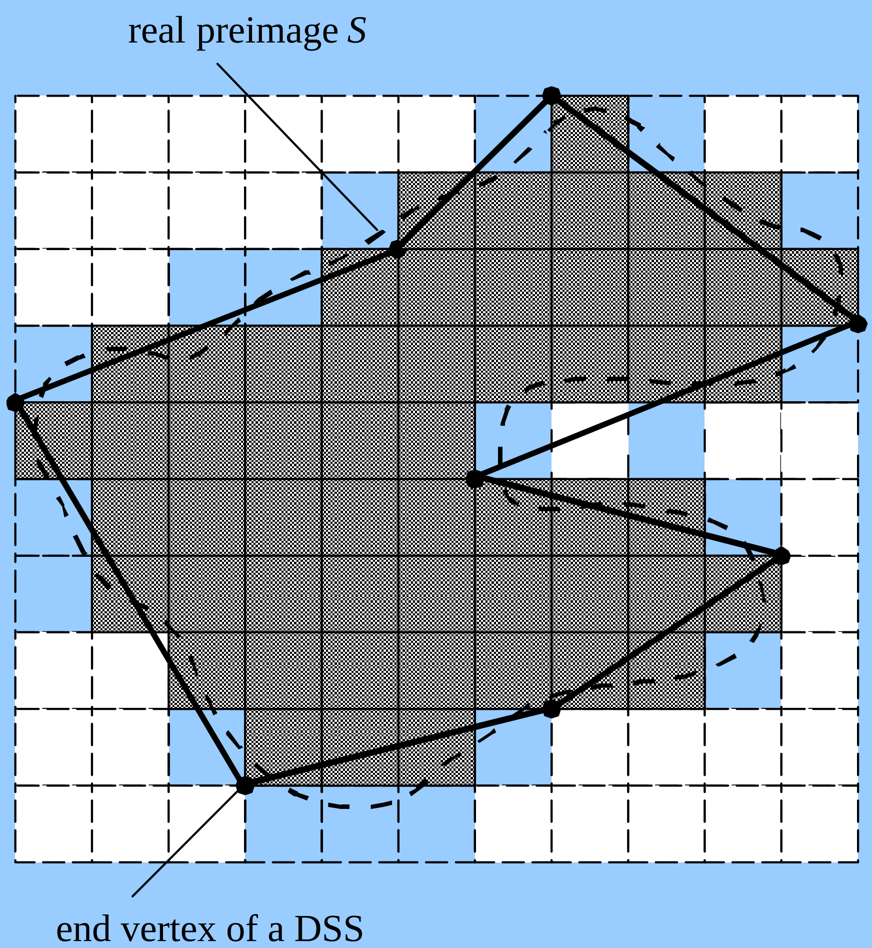


HYPOTHESIS:

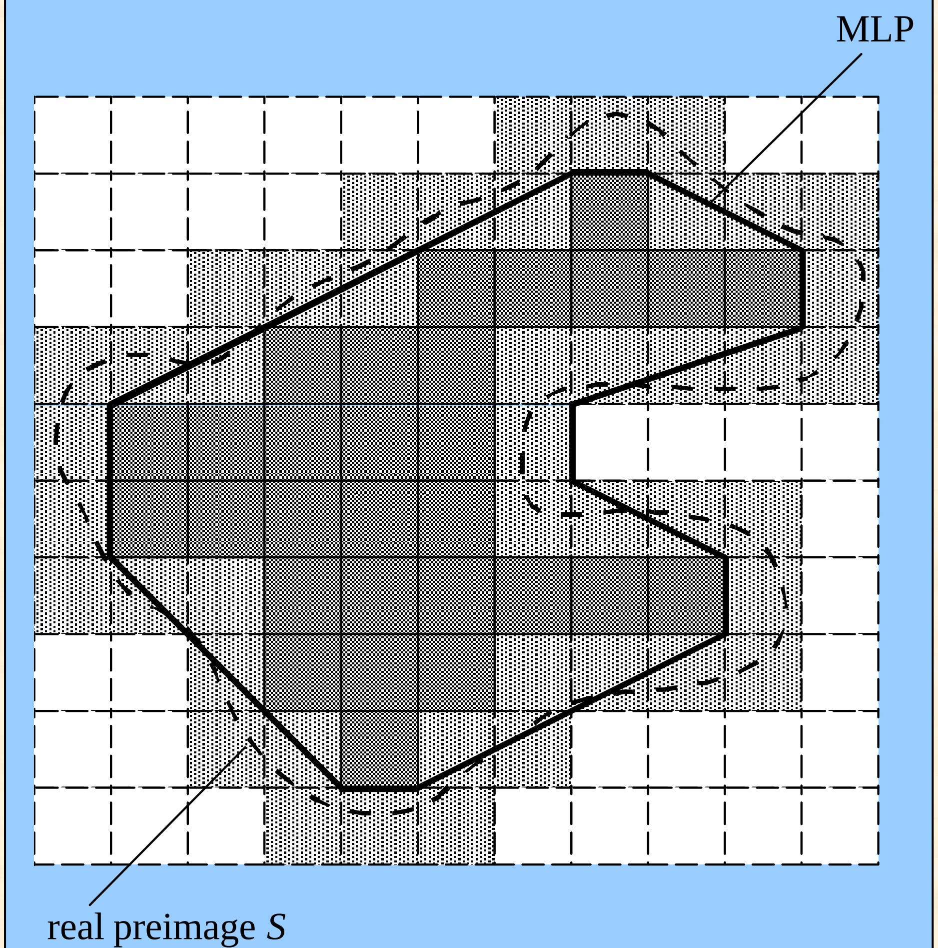
graph-theoretical concepts (with local weights etc.) do **not** allow multigrid convergent length estimation

two multigrid convergent techniques:

DSS approximation



MLP computation



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Convergence Theorems

Theorem: S is a bounded convex set.

Then there exists r_S such that for all $r \geq r_S$

$$|Perimeter(S) - l_r| \leq f(r)$$

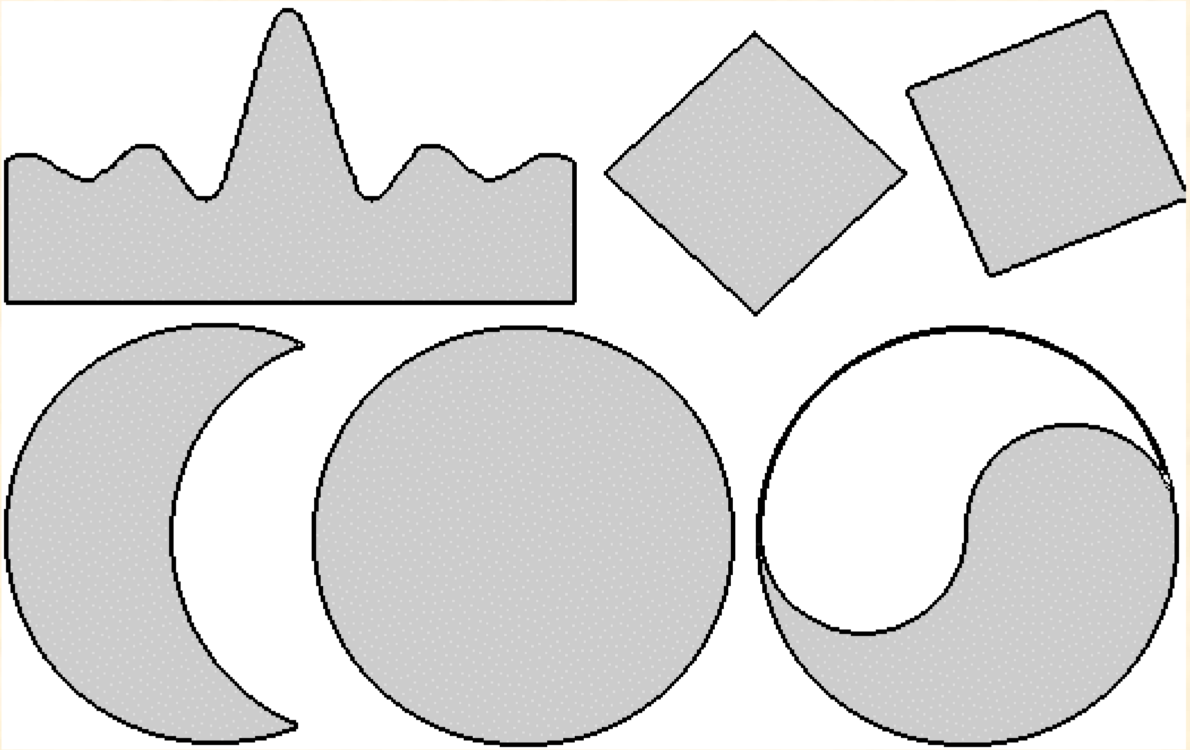
DSS: $f(r) = 4.5 / r$

1992 **W. Kovalevsky/S. Fuchs** 2000 **R. Klette/J. Zunic**

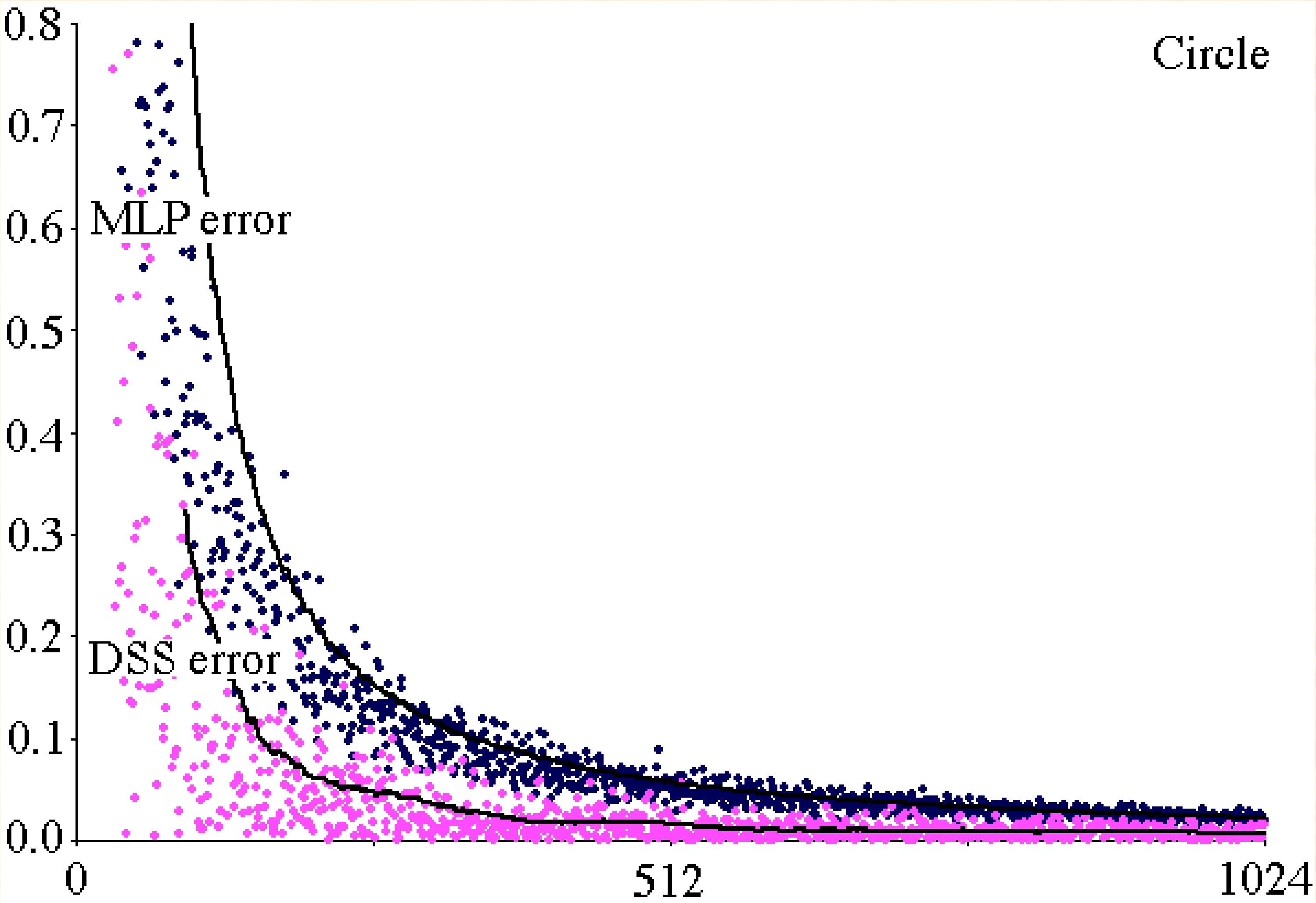
MLP: $f(r) = 8 / r$

1998 **F. Sloboda, B. Zatko**

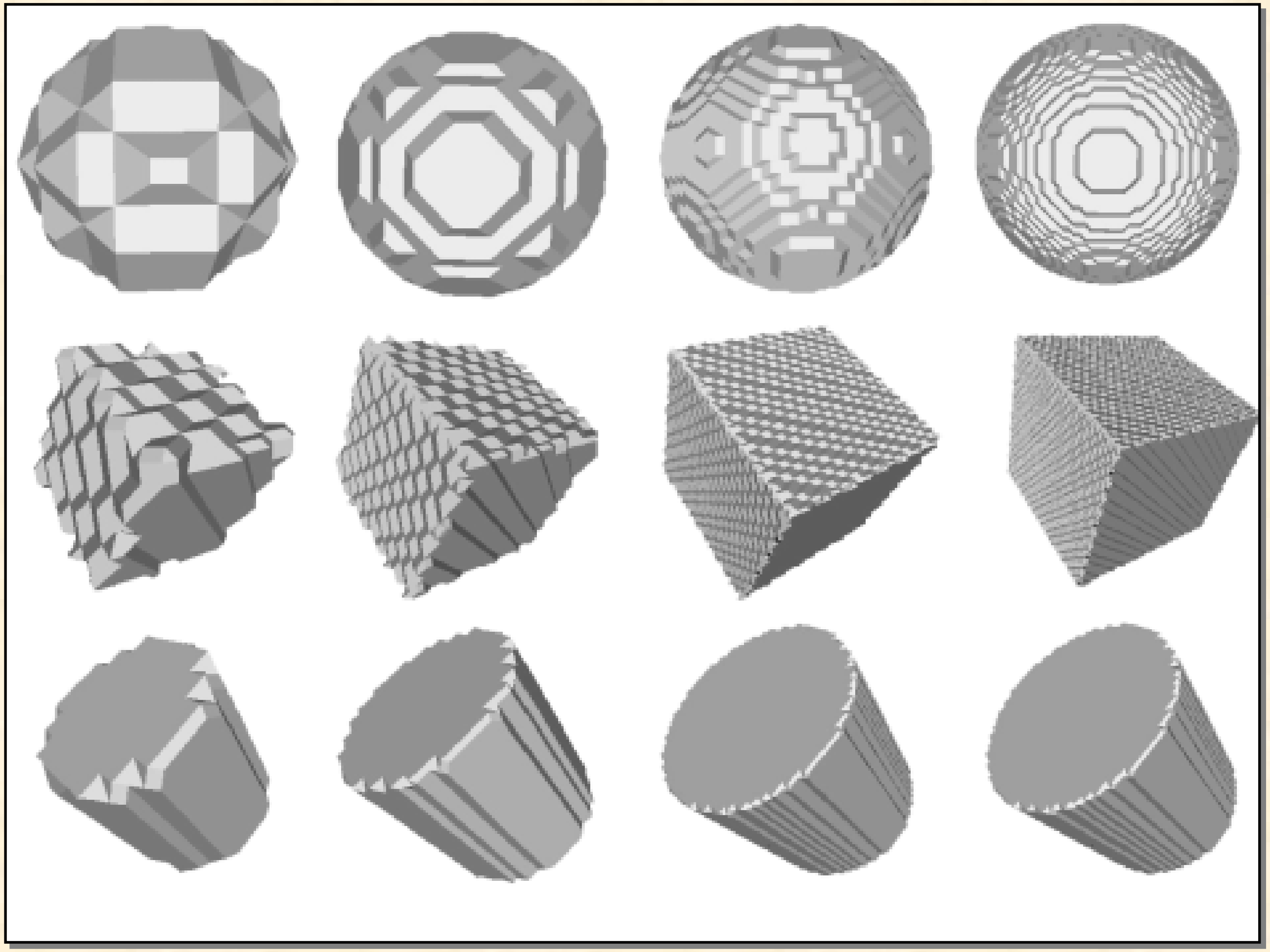
DSS and MLP: Experimental Results



1999 **R. Klette, W. Kovalevsky, B. Yip**


Circle
MLP error
DSS error
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
512
1024

SURFACE AREA IN 3D



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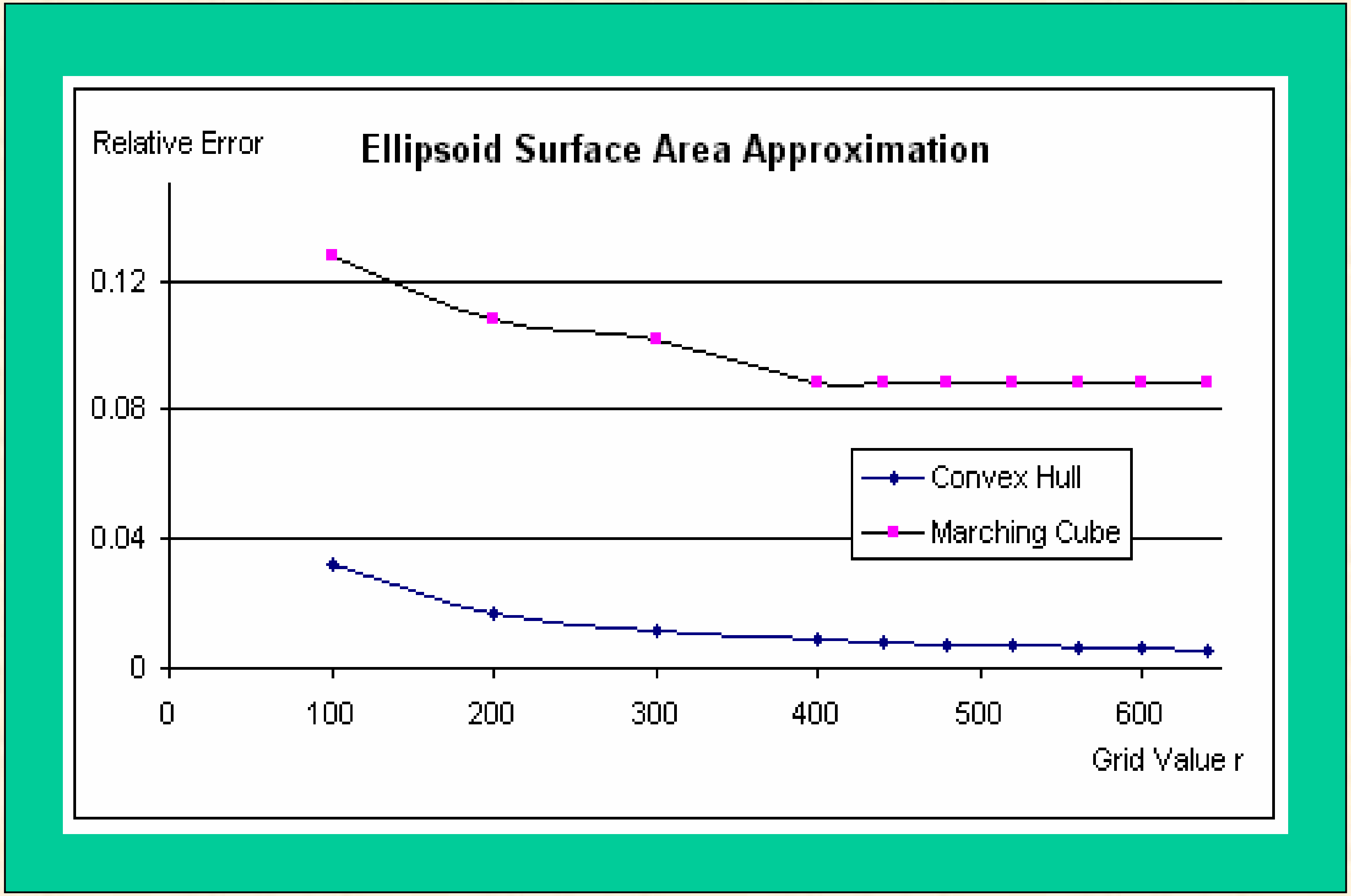

Relative Error
Ellipsoid Surface Area Approximation
0.12
0.08
0.04
0
0
100
200
300
400
500
600
Grid Value r
Convex Hull
Marching Cube

Digital Planar Segment Approximation

a DPS consists of n vertices $\mathbf{p}_i$ satisfying

$$0 \leq \mathbf{n} \cdot \mathbf{p}_i - d < \mathbf{n} \cdot \mathbf{v} \qquad i=1, 2, \ldots, n$$

where $\mathbf{n} \cdot \mathbf{p} = d$ is a Euclidean plane

$\mathbf{v}$ = main diagonal vector of length $\sqrt{3}$

standard plane by

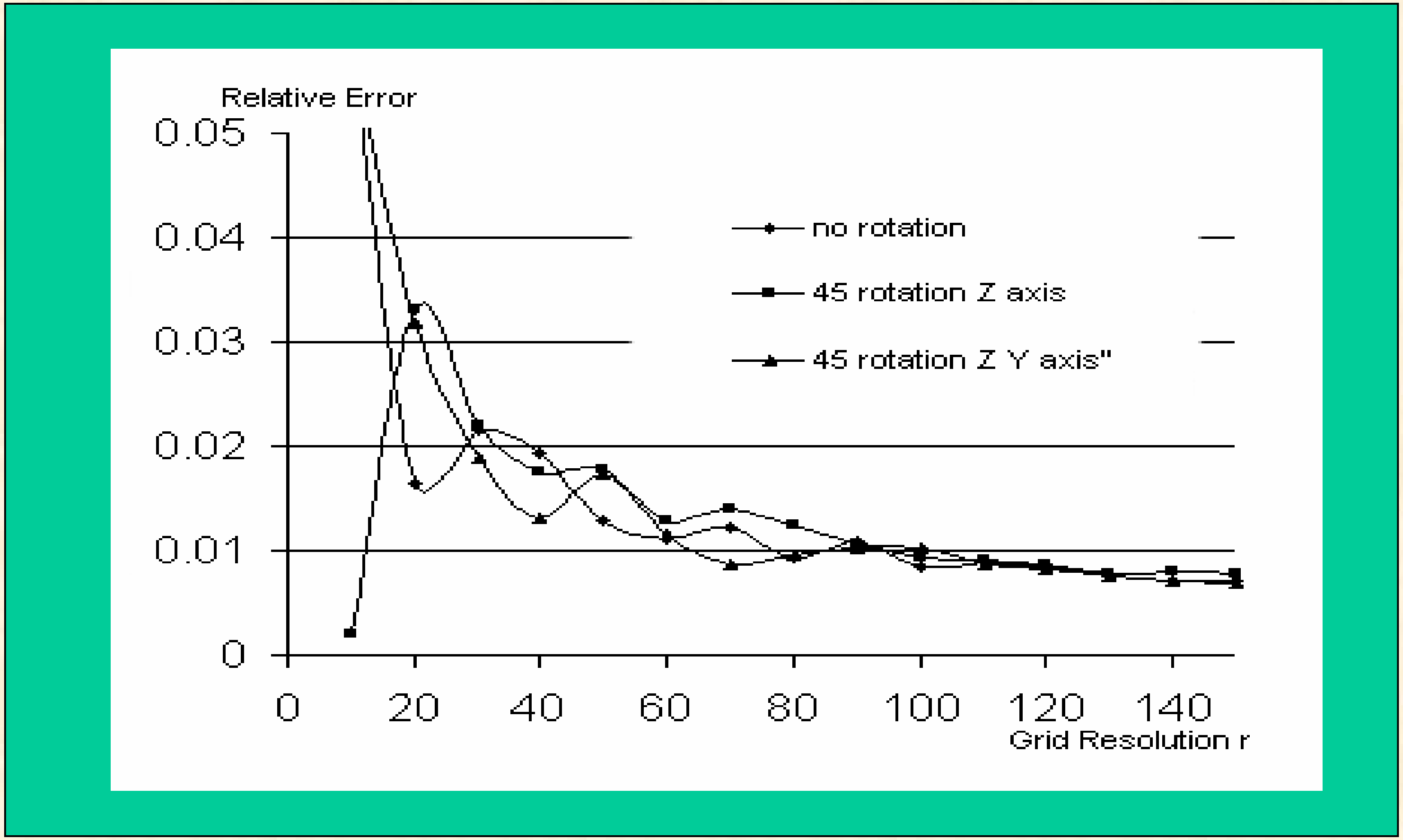
1996 **J.Francon, J.-M.Schramm, M.Tajine**

2000 **R. Klette, H. Sun** - *Incremental Algorithm*

- the surface is traced by using **G.T. Herman's** algorithm and represented by a surface graph
- the problem in each step: given n vertices of a DPS. Still a DPS with $(n+1)$th vertex?
- at each step, a list of all effective supporting planes is maintained. If the list is empty, the vertex set is not a DPS.
- after adding a new vertex, delete those supporting planes no longer effective and construct new effective supporting planes with the added vertex.

Experimental Results

surface area of general ellipsoid at different orientations



relative error at grid resolution 100

marching cube: 10%   **convex hull:** 3.22%   **DPS:** < 1%

2001 **F. Sloboda, B. Zatko** - *Relative convex hull*

$A \subseteq Q \subset R^3$ is *Q-convex* iff for all $p, q \in A$

$pq \subseteq Q$ then $pq \subseteq A$

relative convex hull $CH_Q(P)$ of P with respect to Q

= intersection of all Q-convex sets containing P

Convergence theorem:

S be a compact set in 3D space
bounded by a smooth closed Jordan surface

then $$\lim_{r\to\infty} A\left(CH_{J_r^+(S)}\left(J_r^-(S)\right)\right) = A(S)$$

OPEN PROBLEMS

IN GENERAL

- **sharp bounds for convergence speeds** $f(r)$ **?**
- **optimum convergence speed ?**

SURFACE AREA

- **proof of convergence for DSP method?**
- **feasible algorithm for relative convex hull in 3D**

LENGTH OF A CURVE

- **non-polygonal interpolations for curve digitizations at grid point positions**
- **MLP algorithm in 3D: correctness, time complexity, and convergence speed**


R. Klette
picture taken in 1983