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On the Theory of Inference Operators*

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It is well known that the theory of inductive inference contributes to a theory of interpolation in the discrete case: One investigates algorithms, which, from given values of a function, compute new ones (see **[Go]**, **[Ba]**, **[Fe]**, **[Thi]**, **[Blu]** et al.). In this discrete interpolation theory, such algorithms assign, to certain local informations, constructive, global informations, i.e., one realizes mappings of the form $\Phi: \mathfrak{Z}N^2 \to N$; the values in range Φ (hypotheses) are interpreted in a suitable effective numbering $\beta: N \to P\sigma^1$, where $P\sigma^1$ denotes the set of partially recursive functions. Detailed studies of such inference operators are to be found, e.g., in **[KlLi]**.

At the same time, it is well known that this inductive recognition is a very special case of the one used in mathematical statistics. However, on the one hand, in mathematical statistics one finds hardly any constructive elements (global information, i.e., inductively determined laws mean here parameters of distributions), on the other hand, inductive inference restricts itself a priori to recursive objects (laws mean here computing procedures, Gödel numbers etc.).

Discrete mathematics, however, has already had for quite a while a much more far reaching concept of law, namely that of a constructively described null set.

In this paper, we want to discuss some aspects of a theory of inference operators, in which the used hypotheses are constructively described null sets. For the sake of simplicity, we explain our ideas using the example of the set of indicator functions on N, which we shall identify with the set X^ω ($X = \{0, 1\}$) of infinite sequences with values in X; we assume further $\mu(0) = \mu(1) = \frac{1}{2}$ and denote the product measure on X^ω by $\bar{\mu}$.

Following **[Ma]**, **[Ja]**, **[Sch]**, we call a set $\mathfrak{N} \subseteq X^\omega$ a recursive null set, if there

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exists a recursive sequential test for $\mathfrak{N}$, i.e., a Y with the following properties:

(R1) $Y \subseteq N \times X^*$ and Y is enumerable.

(R2) If we let $[Y_i] =_{\text{def}} \{p: [i, p] \in Y\} \cdot X^\omega$, then $\bar{\mu}([Y_i]) \leq 2^{-i}$.

(R3) If we let $\mathfrak{N}_Y =_{\text{def}} \bigcap_i [Y_i]$, then $\mathfrak{N}_Y = \mathfrak{N}$.

Moreover, following [Sch], we shall call $\mathfrak{N}$ a totally recursive null set, if there exists a recursive sequential test Y for $\mathfrak{N}$, such that the following holds:

(T4) The function f with $f(i) =_{\text{def}} \bar{\mu}([Y_i])$ is computable in the sense of recursive analysis.

What are the general implications, if one pursues inductive recognition by recursive null sets, i.e., if one considers inference operators $\Phi: X^\omega \to N$, for which $h \in$ range Φ is interpreted by an effective numbering β of recursive sequential tests (null sets)?

For this, we let $AKZ(\Phi, \beta) =_{\text{def}} \{\xi: \xi \in \text{domain } \Phi \wedge \xi \in \beta \circ \Phi(\xi)\}$ ("$\in$" is here shorthand notation for: ξ is an element of the null set, described by the sequential test $\beta \circ \Phi(\xi)$).

(I) The results of the theory of inductive recognition carry over. (According to [Sch], given any recursive $\xi \in X^\omega$, there is a totally recursive sequential test Y with $\mathfrak{N}_Y = \{\xi\}$.)

(II) The types of realizations of Φ are very similar to those discussed in inductive inference. (Identification by numbering, limit recursion, etc.; see [So], [Go].)

(III) The weaker concept of law allows also the nontrivial recognition of certain nonrandom and nonrecursive objects.

(IV) The fusion of statistics and constructive mathematics, seen as a goal by [Sch], is completed.

In order to illustrate (I)–(IV) we formulate here some statements about a certain type of recognition, using regular null sets. As in [McN], we call $M \subseteq X^\omega$ regular, if there exist regular $W_i, V_i \subseteq X^*$ $(1 \leq i \leq n)$ such that $M = \bigcup_{i=1}^n W_i V_i^\omega$. Since recursive null sets are, topologically speaking, G_δ sets (i.e., countable intersections of open sets), we shall consider from now on regular G_δ sets. From [StWa] we know

LEMMA 1. *The following statements are equivalent:*

(a) *M is a regular G_δ set;*

(b) *there are regular $W_i, V_i \subseteq X^*$ $(1 \leq i \leq n)$, which are totally disordered with respect to the initial word relation such that $M = \bigcup_{i=1}^n W_i V_i^\omega$;*

(c) *there is a finitely determined acceptor $\mathfrak{A} = [Z, X, f, z_0]$ and a $\mathfrak{Z} \subset \mathfrak{P}Z$, such that*

$$M = \{\xi: \exists Z' \, (Z' \in \mathfrak{Z} \wedge U(\Phi_{\mathfrak{A}}(\xi)) \cap Z' \neq \emptyset)\}$$

where $\Phi_{\mathfrak{A}}(\xi) =_{\text{def}} f(z_0, \xi(1)) \cdot \cdots \cdot f(z_0, \xi(1) \cdots \xi(n))$ *and* $U(\zeta) = \{z: \text{card}\{n: \zeta(n) = z\} = \aleph_0\}$.

This immediately implies

THEOREM 2. *Every regular G_δ null set is totally recursive.*

Let R be the family of all regular G_δ null sets and $\mathfrak{N}_R =_{\text{def}} \bigcup R$; let Δ be an effective numbering of all recursive null sets (which exists according to [Ma], [Sch]). For any generally recursive function $F: X^* \to N$, we shall consider the set

$$(*) \qquad \Delta N(F) \underset{\text{def}}{=} \left\{ \xi : s = \lim_{p \to \xi} F(p) \wedge \Delta(s) \in R \wedge \xi \in \mathfrak{C}(\Delta(s)) \right\}$$

of all "laws", recognized by F, by a limit procedure. (Here $\mathfrak{C}(M)$ denotes the closure of M.) This definition was made, following the method "GN" in [Ba], for example.

Definition (*) does not force us to leave the regular G_δ null sets because of

LEMMA 3 (STAIGER). *$\mathfrak{C}(M) \in R$ for $M \in R$.*

Further let $\Delta N =_{\text{def}} \{\Delta N(F) : F$ generally recursive$\}$ and $\mathfrak{N}_{\Delta N} =_{\text{def}} \bigcup \Delta N$.

COROLLARY 4. $\mathfrak{N}_R = \mathfrak{N}_{\Delta N}$.

Through identification by enumeration one has

THEOREM 5. *Given any effective $\beta: N \to R$, one can find effectively a generally recursive F such that $\mathfrak{N}_\beta \subseteq \Delta N(F)$ for $\mathfrak{N}_\beta =_{\text{def}} \bigcup_n \beta_n$.*

In the proof one uses the fact that the property $p \cdot X^\omega \cap M \neq \varnothing$, for $p \in X^*$ and regular $M \subseteq X^\omega$ can be decided. This, too, explains why we restrict ourselves here to regular null sets.

Let $\mathfrak{N}_T$ be the union of all totally recursive null sets. A comparison with $\mathfrak{N}_{\Delta N}$ gives

THEOREM 6. $\mathfrak{N}_T \supset \mathfrak{N}_{\Delta N}$.

One can prove the stronger result: If $\xi = 01 \cdots 0^n 1^n \cdots$ is an element of a regular set $M \subseteq X^\omega$, then $\bar{\mu}(M) > 0$.

Because of Corollary 4 we have, therefore, for the set $Al \subseteq X^\omega$ of generally recursive sequences

THEOREM 7. *$Al \setminus \mathfrak{N}_{\Delta N} \neq \varnothing$, since it is well known that $Al \subset \mathfrak{N}_T$ (see [Sch]).*

Thus, on the one hand, one has recognitions of type ΔN for nonrecursive, non-random sequences; on the other hand, not all recursive sequences are ΔN-recognizable.

We ask ourselves now whether there exists a universal ΔN-recognition for $\mathfrak{N}_R$. To this effect, we prove

THEOREM 8. *The property of a regular G_δ set to be a null set is decidable.*

By Lemma 1, it suffices to decide the property $\bar{\mu}([V]) < 1$ for $V \subseteq X^*$ regular and totally disordered.

To this end we use

LEMMA 9 (PAZ, WECHSUNG). *If $V \subseteq X^*$ is totally disordered and if V is accepted by the finite determined automaton $\mathfrak{A} = [X, Z, f, z_0, F]$ then one can effectively find polynomials g and h of degree* card Z *with rational coefficients, such that $\bar{\mu}([V]) = g(2^{-1}) \cdot (h(2^{-1}))^{-1}$.*

Thus one has

Corollary 10. *One can construct an effective* $\beta: N \to R$ *such that* $\beta(N) = R$.

Hence we conclude from Theorem 5 and Lemma 3

Theorem 11. *One can construct a universal generally recursive* $F: X^* \to N$ *such that* $\Delta N(F) = \mathfrak{N}_R$.

At the same time one has

Theorem 12. *One can construct a totally recursive sequential test Y such that* $\mathfrak{N}_R \subseteq \mathfrak{N}_Y$.

For the proof one uses the following lemma, which goes back to [Sch], because of Theorem 2.

Lemma 13. *Given any effective* $\beta: N \to R$, *one can find a totally recursive sequential test Y such that* $\mathfrak{N}_\beta \subseteq \mathfrak{N}_Y$.

This is another confirmation of Theorem 6, if one uses

Theorem 14 (Schnorr). *Given any totally recursive test Y, one can effectively construct a* $\xi \in Al \setminus \mathfrak{N}_Y$.

Since there exists an effective numbering of all acceptors of regular G_δ null sets such that the state numbers of the numbered automata increase, Theorem 8 implies the following sharpening of Corollary 10.

Theorem 15. *One can give an effective Occam numbering* $\hat{\beta}: N \to R$ *for R.*

For the strategy $F: X^* \to N$, constructed in Theorem 11, we even get now that $\Delta \circ \Phi(\xi)$ is a weight-minimal regular G_δ null set for every $\xi \in \mathfrak{N}_R$. By this, complexity in the discovered laws becomes accessible to investigation.

The theorems formulated here for the case of ΔN-realizations of inference operators confirm sufficiently (I)—(IV) and suggest how to continue the investigations for other types of realizations (GH', GN^∞, etc.; see [**Ba**]) and for other classes of null sets constructively described. This will be done in a subsequent paper which is being prepared.

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