Gradient Space	Light Source and Lambertian Reflectance	3PSM	Integration of Gradient Fields

Pixel Labeling: Photometric Stereo¹

Lecture 20

See Material in Section 7.4 Reinhard Klette: Concise Computer Vision Springer-Verlag, London, 2014

¹See last slide for copyright information.

Photometric Stereo Method

Stereo Vision: one light source and multiple static or mobile cameras

Now: one static camera and multiple static light sources (indoor)

Light sources are switched on and off while taking images

Defines photometric stereo method (PSM), example of shape from shading

We describe

- Gradient space (for modelling reflectance maps)
- **2** Lambertian reflectance (e.g. human skin is approximate Lambertian)
- **3** 3-light-source PSM (3PSM) for deriving surface gradients
- **4** How surface gradients can be mapped into a 3D shape

3PSM Example



Three input images for PSM; the hand is not allowed to move when taking these images. Reconstructed surface (in 1994) by 3PSM

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Agenda			

1 Gradient Space

2 Light Source and Lambertian Reflectance

3 3PSM

Integration of Gradient Fields

Surface Gradients and Normals

Gradient of surface Z = f(X, Y) in 3D space

.

$$\nabla Z = \mathbf{grad} \ Z = \left[\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y} \right]^{\top}$$

Example: A plane Z = aX + bY + c, then gradient $[a, b]^{\top}$

Normal of surface Z = f(X, Y)

$$\mathbf{n} = \left[\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}, 1\right]^{\top} = [a, b, 1]^{\top}$$

Going away from the image plane by having value +1 in third component We use of *a* and *b* for denoting X- and Y-components of a normal

Unit normal

Unit normal (of length 1)



Gaussian sphere: radius 1, centered at origin O

Slant σ : angle between vector \mathbf{n}° and Z-axis Tilt θ : angle between vector from O to P = (X, Y, 0) and X-axis Unit normal \mathbf{n}° uniquely represented by spherical coordinates (σ, θ) *ab*-space of gradient or normal coordinates *a* and *b* Point (a, b) in gradient space represents gradient $[a, b]^{\top}$ in *XYZ* space

Example

Plane Z = aX + bY + c in XYZ space

(a,b) in gradient space represents all planes parallel to the given plane (i.e., for any $c\in\mathbb{R}$)

Orthogonal Normals

Given: $\mathbf{n}_1 = [a_1, b_1, 1]^\top$ in *XYZ* space (say: direction to a light source) **Task:** characterize normal $\mathbf{n}_2 = [a_2, b_2, 1]^\top$ orthogonal to \mathbf{n}_1 (say: directions where surface points are "just" not illuminated anymore)

Dot product of both vectors:

$$\mathbf{n_1} \cdot \mathbf{n_2} = a_1 a_2 + b_1 b_2 + 1 = ||\mathbf{n_1}||_2 \cdot ||\mathbf{n_2}||_2 \cdot \cos \frac{\pi}{2} = 0$$

Given a_1 and b_1 define a straight line $\gamma : a_1a_2 + b_1b_2 + 1 = 0$ in gradient space with unknowns a_2 and b_2 :

() the line incident with origin o = (0,0) and point p is orthogonal to line γ

2
$$\sqrt{a_1^2 + b_1^2} = d_2((a_1, b_1), o) = 1/d_2(p, o)$$

3 p and (a_1, b_1) are in opposite quadrants

Line $\boldsymbol{\gamma}$ is uniquely defined by these three properties on page before

Line γ is called the *dual straight line* to normal \mathbf{n}_1 or to (a_1, b_1)

Any direction \mathbf{n}_2 orthogonal to \mathbf{n}_1 is located on γ



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Point-Light Source Assumptions

- $\textbf{0} A \text{ light source } L \text{ is identified with a single point in } \mathbb{R}^3$
- 2 Light source emits light uniformly in all directions; with intensity

EL

(i.e. integral over energy distribution curve of light source L in visible spectrum of light)

 Light source "not very close" to objects of interest (objects are "small" compared to the distance to the light source); in direction

$$\mathbf{s}_L = [\mathbf{a}_L, \mathbf{b}_L, 1]^ op$$

relatively to illuminated surfaces

Model is abstract: an existing light source actually

Illuminates only a limited range It has a particular energy distribution curve $L(\lambda)$ It has a geometric shape

Illuminated and Recorded Point on a Surface

Surface normal **n**, direction **s** to light source, viewing directions \mathbf{v}_i



Lambertian reflector: Radiant intensity observed from an diffusely reflecting surface is directly proportional to the cosine of angle α (i.e. viewing direction does not matter as long as point is visible)

Recall Inner Vector Product and Albedo

Normal $\mathbf{n}_P = [a, b, 1]^{\top}$ at a visible and illuminated surface point P

1 Inner vector product:

$$\mathbf{s}^{\top} \cdot \mathbf{n}_{P} = ||\mathbf{s}||_{2} \cdot ||\mathbf{n}_{P}||_{2} \cdot \cos \alpha$$

defines

$$\cos \alpha = \frac{\mathbf{s}^{\top} \cdot \mathbf{n}_{P}}{||\mathbf{s}||_{2} \cdot ||\mathbf{n}_{P}||_{2}}$$

2 Emitted light at a point P is scaled by

$$\eta(P) = \rho(P) \cdot \frac{E_L}{\pi}$$

where π is the spatial angle of a halfsphere $\rho(P)$ is the *albedo* (i.e. surface reflectance) at *P*, with $0 \le \rho(P) \le 1$

Lambert's Cosine Law

Emitted light at surface point P, called the *reflectance* at P:

$$R(P) = \eta(P) \cdot \frac{\mathbf{s}_{L}^{\top} \cdot \mathbf{n}_{P}}{||\mathbf{s}_{L}||_{2} \cdot ||\mathbf{n}_{P}||_{2}}$$
$$= \eta(P) \cdot \frac{a_{L}a + b_{L}b + 1}{\sqrt{a_{L}^{2} + b_{L}^{2} + 1} \cdot \sqrt{a^{2} + b^{2} + 1^{2}}}$$

Reflectance $R(P) \ge 0$ is a second-order curve in unknowns *a* and *b* $\eta(P)$ depends on energy E_L and albedo $\rho(P)$

Reflectance Maps

A *reflectance map* is defined in *ab* gradient space

Reflectance map assigns a reflectance value to gradient value (a, b) assuming that the given surface reflectance is uniquely defined by the gradient value (as it is the case for Lambertian reflectance)

Lambertian Reflectance Map:

Point P with albedo $\rho(P)$ and gradient (a, b)

1 $n_P = s_L$:

 $\alpha = 0$ and $\cos \alpha = 1$; curve degenerates into $R(P) = \eta(P)$, the maximal possible value; value $\eta(P)$ at point (a_L, b_L) in gradient space

2 n_P orthogonal to s_L:

surface point *P* is "just" not illuminated anymore; \mathbf{n}_P is on the dual straight line to \mathbf{s}_L ; $\alpha = \pi/2$ and $\cos \alpha = 0$; curve degenerates into R(P) = 0, the minimal possible value; value 0 along the straight line dual to (a_L, b_L) in gradient space

Isolines in a Lambertian Reflectance Map



Duality between straight line and direction to light source

For R(P) between 0 and $\eta(P)$, the curve is either parabolic or hyperbolic

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1 Gradient Space

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3 3 PSM

Integration of Gradient Fields

Recovering Surface Gradients

One static camera (on a tripod) and three different point-light sources at directions \mathbf{s}_i , for i = 1, 2, 3

Reflectance map value $R_i(P)$ uniformly scaled by constant c > 0 (due to distance between object and camera) and mapped into monochromatic value u_i in the image

Capture three images, turn only one light source on at a time

Point P maps at the same pixel position into three intensity values

$$u_i = R_i(P) = \frac{\eta(P)}{c} \cdot \frac{\mathbf{s}_i^\top \cdot \mathbf{n}_P}{||\mathbf{s}_i||_2 \cdot ||\mathbf{n}_P||_2}$$

defining three second-order curves in gradient space, which (ideally) intersect at (a, b), where $\mathbf{n}_P = [a, b, 1]^{\top}$ is the normal at point P

${\sf Albedo-Independent}\ {\sf PSM}$

Consider surfaces with (approximate) Lambertian reflectance We cannot assume that albedo $\rho(P)$ is constant on a recorded surface Coloring of surface points changes and thus also the albedo We need to consider *albedo-independent PSM*

Example:

Human skin is approximate Lambertian but with changes in albedo Non-Lambertian: e.g. specularities in open eyes

Derivation of an Algebraic Solution

We have three equations

$$u_i = E_i \cdot \frac{\rho(P)}{c\pi} \cdot \frac{\mathbf{s}_i^\top \cdot \mathbf{n}_P}{||\mathbf{s}_i||_2 \cdot ||\mathbf{n}_P||_2}$$

for three light sources, with i = 1, 2, 3

- **1** Multiply first equation (i.e. i = 1) by $u_2 \cdot ||\mathbf{s}_2||_2$
- 2 Multiply second equation (i.e. i = 2) by $-u_1 \cdot ||\mathbf{s}_1||_2$
- 3 Add both; this results into

$$\rho(P) \cdot \mathbf{n}_P \cdot (E_1 u_2 || \mathbf{s}_2 ||_2 \cdot \mathbf{s}_1 - E_2 u_1 || \mathbf{s}_1 ||_2 \cdot \mathbf{s}_2) = 0$$

Thus: vector \mathbf{n}_P is orthogonal to this difference of vectors \mathbf{s}_1 and \mathbf{s}_2 **4** Assume $\rho(P) > 0$. If using Images 1 and 3, we have that

$$\mathbf{n}_{P} \cdot (E_1 u_3 || \mathbf{s}_3 ||_2 \cdot \mathbf{s}_1 - E_3 u_1 || \mathbf{s}_1 ||_2 \cdot \mathbf{s}_3) = 0$$

 \mathbf{n}_{P} is collinear with the cross product

$$(E_1u_2||\mathbf{s}_2||_2 \cdot \mathbf{s}_1 - E_2u_1||\mathbf{s}_1||_2 \cdot \mathbf{s}_2) \times (E_1u_3||\mathbf{s}_3||_2 \cdot \mathbf{s}_1 - E_3u_1||\mathbf{s}_1||_2 \cdot \mathbf{s}_3)$$

This uniquely defines the unit normal $\mathbf{n}_{\mathcal{P}}^{\circ}$ pointing away from the camera

Needed:

- Directions to light sources
- 2 Relative intensities of light sources (no absolute measurements E_i)

Calibration Sphere



Calibration sphere: Uniform albedo and Lambertian reflectance

Use mean intensities in recorded calibration sphere images for estimating energy ratios between intensities E_i of the three light sources

Illustration of detected iso-intensity "lines" with expected noisiness

Direction to Light Sources by Inverse PSM

Calibration of directions s_i to the three light sources

We apply inverse photometric stereo:

Use of a Lambertian calibration sphere with uniform albedo

Place sphere about at location of objects (to be modelled)

- 1 Identify the circular border of the imaged sphere
- 2 Calculate surface normals (of the sphere) at more than three points P (say, at about 100) projected into pixel positions within the circle How?
- 3 Identify direction s_i by least-square error optimization using the 3PSM solution equations
- **4** We have values u_i and normals \mathbf{n}_P ; solve for unknown direction \mathbf{s}_i

Example: Human Faces

3PSM is of reasonable accuracy for recovering the albedo values of a human face



Face recovered by 3PSM (at University of Auckland in 2000) Closed eyes avoid the recording of specularity

Gradient Space	Light Source and Lambertian Reflectance	3PSM	Integration of Gradient Fields
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4 Integration of Gradient Fields

Discrete Gradient Field

Integration: Discrete fields of gradients into a surface

Integration is not unique even when dealing with smooth surfaces

Result only determined up to an additive constant

III-Posedness of Discrete Integration

Results of PSM are discrete and erroneous surface gradient data

Surfaces often "non-smooth" (e.g. polyhedral)

Example: Camera looks onto a stair case, orthogonal to the front faces; recovered normals point straight towards camera

Densities of recovered surface normals do not correspond uniformly to local surface slopes

Global Integration

Gradient vector estimated at any $p \in \Omega$

Task: map this uniform and dense gradient vector field into a surface in 3D space which is likely to be the actual surface which caused the estimated gradient vector field

Depth values Z(x, y) define labels at pixel locations (x, y)

Back to a labeling problem with error (or energy) minimization

Data term

$$E_{data}(Z) = \sum_{\Omega} [(Z_x - a)^2 + (Z_y - b)^2] + \lambda_0 \sum_{\Omega} [(Z_{xx} - a_x)^2 + (Z_{yy} - b_y)^2]$$

Smoothness term

$$E_{smooth}(Z) = \lambda_1 \sum_{\Omega} [Z_x^2 + Z_y^2] + \lambda_2 \sum_{\Omega} [Z_{xx}^2 + 2Z_{xy}^2 + Z_{yy}^2]$$

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Notation			

 Z_x and Z_y

first-order partial derivatives of Z

 a_x and b_y

first-order partial derivatives of a and b

 Z_{xx} , $Z_{xy} = Z_{yx}$, and Z_{yy}

second-order partial derivatives of Z

 $\lambda_0 \geq 0$ controls consistency between surface curvature and changes in available gradient data

 $\lambda_1 \geq 0$ controls smoothness of surface

 $\lambda_2 \geq 0$ controls smoothness of surface curvature

Total Energy, MRF Model, and Two Algorithms

Determine surface Z (i.e. the labeling function) such that total error (or total energy)

$$E_{total}(Z) = E_{data}(Z) + E_{smooth}(Z)$$

is minimized; derivatives in E_{smooth} define MRF again

Two Algorithms

Frankot-Chellappa algorithm is for $\lambda_0 = \lambda_1 = \lambda_2 = 0$, thus not using the second part of the data constraint and no smoothness constraint at all

Wei-Klette algorithm also uses second-order derivatives (curvature) and smoothness optimization

Optimization problem can be solved by using the theory of projections onto convex sets

Gradient field $(a_{x,y}, b_{x,y})$ is projected onto the nearest integrable gradient field in the least-square sense, using the Fourier transform for optimizing in the frequency domain

2D DFT of surface function Z(x, y)

$$\mathbf{Z}(u,v) = \frac{1}{|\Omega|} \sum_{(x,y)\in\Omega} Z(x,y) \cdot \exp\left[-i2\pi\left(\frac{xu}{N_{cols}} + \frac{yv}{N_{rows}}\right)\right]$$

Inverse transform

$$Z(x,y) = \sum_{(u,v)\in\Omega} \mathbf{Z}(u,v) \cdot \exp\left[i2\pi\left(\frac{xu}{N_{cols}} + \frac{yv}{N_{rows}}\right)\right]$$

 $i = \sqrt{-1}$ and u and v represent frequencies in 2D Fourier domain

Relevant Fourier Pairs and Parseval's Theorem

$$\begin{array}{rcl} Z_{x}(x,y) & \Leftrightarrow & iu \ \mathbf{Z}(u,v) \\ Z_{y}(x,y) & \Leftrightarrow & iv \ \mathbf{Z}(u,v) \\ Z_{xx}(x,y) & \Leftrightarrow & -u^{2} \ \mathbf{Z}(u,v) \\ Z_{yy}(x,y) & \Leftrightarrow & -v^{2} \ \mathbf{Z}(u,v) \\ Z_{xy}(x,y) & \Leftrightarrow & -uv \ \mathbf{Z}(u,v) \end{array}$$

Define the appearance of considered derivatives of Z in frequency domain $\mathbf{A}(u, v)$ and $\mathbf{B}(u, v)$ be Fourier transforms of gradients $A(x, y) = a_{x,y}$ and $B(x, y) = b_{x,y}$, respectively

Parseval's Theorem

$$\frac{1}{|\Omega|}\sum_{\Omega} |Z(x,y)|^2 = \sum_{\Omega} |\mathbf{Z}(u,v)|^2$$

Optimization in Frequency Domain

In conclusion to Parseval's Theorem: Equivalence of optimization problem in spatial domain to optimization problem in frequency domain

Minimize, where sums are for $(u, v) \in \Omega$:

$$\sum_{\Omega} \left[(iu\mathbf{Z} - \mathbf{A})^2 + (iv\mathbf{Z} - \mathbf{B})^2 \right]$$

+ $\lambda_0 \sum_{\Omega} \left[(-u^2\mathbf{Z} - iu\mathbf{A})^2 + (-v^2\mathbf{Z} - iv\mathbf{B})^2 \right]$
+ $\lambda_1 \sum_{\Omega} \left[(iu\mathbf{Z})^2 + (iv\mathbf{Z})^2 \right]$
+ $\lambda_2 \sum_{\Omega} \left[(-u^2\mathbf{Z})^2 + 2(-uv\mathbf{Z})^2 + (-v^2\mathbf{Z})^2 \right]$

Start of Solution Process

Above expression expanded into

$$\sum_{\Omega} \left[u^{2} \mathbf{Z} \mathbf{Z}^{*} - iu \mathbf{Z} \mathbf{A}^{*} + iu \mathbf{Z}^{*} \mathbf{A} + \mathbf{A} \mathbf{A}^{*} \right. \\ \left. + v^{2} \mathbf{Z} \mathbf{Z}^{*} - iv \mathbf{Z} \mathbf{B}^{*} + iv \mathbf{Z}^{*} \mathbf{B} + \mathbf{B} \mathbf{B}^{*} \right] \\ \left. + \lambda_{0} \sum_{\Omega} \left[u^{4} \mathbf{Z} \mathbf{Z}^{*} - iu^{3} \mathbf{Z} \mathbf{A}^{*} + iu^{3} \mathbf{Z}^{*} \mathbf{A} + u^{2} \mathbf{A} \mathbf{A}^{*} \right. \\ \left. + v^{4} \mathbf{Z} \mathbf{Z}^{*} - iv^{3} \mathbf{Z} \mathbf{B}^{*} + iv^{3} \mathbf{Z}^{*} \mathbf{B} + v^{2} \mathbf{B} \mathbf{B}^{*} \right] \\ \left. + \lambda_{1} \sum_{\Omega} \left(u^{2} + v^{2} \right) \mathbf{Z} \mathbf{Z}^{*} \right. \\ \left. + \lambda_{2} \sum_{\Omega} \left(u^{4} + 2u^{2}v^{2} + v^{4} \right) \mathbf{Z} \mathbf{Z}^{*} \right]$$

 \star denotes the complex conjugate, and sums for $(u,v)\in\Omega$

Optimization in Frequency Space

Differentiating the above expression with respect to Z^* and setting the result to zero, we can deduce the necessary condition for a minimum of the cost function

For each $(u, v) \in \Omega$ we have

$$\begin{pmatrix} u^{2}\mathbf{Z} + iu\mathbf{A} + v^{2}\mathbf{Z} + iv\mathbf{B} \end{pmatrix} + \lambda_{0} \left(u^{4}\mathbf{Z} + iu^{3}\mathbf{A} + v^{4}\mathbf{Z} + iv^{3}\mathbf{B} \right)$$

+ $\lambda_{1} \left(u^{2} + v^{2} \right) \mathbf{Z} + \lambda_{2} \left(u^{4} + 2u^{2}v^{2} + v^{4} \right) \mathbf{Z} = 0$

A rearrangement of this equation yields

$$\begin{bmatrix} \lambda_0 (u^4 + v^4) + (1 + \lambda_1) (u^2 + v^2) + \lambda_2 (u^2 + v^2)^2 \end{bmatrix} \mathbf{Z}(u, v) + i (u + \lambda_0 u^3) \mathbf{A}(u, v) + i (v + \lambda_0 v^3) \mathbf{B}(u, v) = 0$$

Gradient Space	Light Source and Lambertian Reflectance	3PSM	Integration of Gradient Fields
Solution			

Solve the above equation except for $(u, v) \neq (0, 0)$:

$$\mathbf{Z}(u,v) = \frac{-i(u+\lambda_0 u^3) \mathbf{A}(u,v) - i(v+\lambda_0 v^3) \mathbf{B}(u,v)}{\lambda_0 (u^4 + v^4) + (1+\lambda_1) (u^2 + v^2) + \lambda_2 (u^2 + v^2)^2}$$

Result

This is the Fourier transform of the unknown surface function Z(x, y) expressed as a function of the Fourier transforms of the given gradients $A(x, y) = a_{x,y}$ and $B(x, y) = b_{x,y}$

Algorithm Part 1: Forward Transform

- 1: **input** gradients a(x, y), b(x, y); parameters λ_0, λ_1 , and λ_2
- 2: for $(x, y) \in \Omega$ do
- 3: if $(|a(x, y)| < c_{\max} \& |b(x, y)| < c_{\max})$ then
- 4: A1(x,y)=a(x,y); A2(x,y)=0;
- 5: B1(x,y)=b(x,y); B2(x,y)=0;
- 6: **else**
- 7: A1(x,y)=0; A2(x,y)=0;
- 8: B1(x,y)=0; B2(x,y)=0;
- 9: end if
- 10: end for
- 11: Calculate Fourier transform in place: A1(u,v), A2(u,v);
- 12: Calculate Fourier transform in place: B1(u,v), B2(u,v);

Algorithm Part 2: Optimize in Frequency Domain

1: for
$$(u, v) \in \Omega$$
 do
2: if $(u \neq 0 \& v \neq 0)$ then
3: $\Delta = \lambda_0 (u^4 + v^4) + (1 + \lambda_1) (u^2 + v^2) + \lambda_2 (u^2 + v^2)^2$;
4: $H1(u, v) = [(u + \lambda_0 u^3)A2(u, v) + (v + \lambda_0 v^3)B2(u, v)]/\Delta$;
5: $H2(u, v) = [-(u + \lambda_0 u^3)A1(u, v) - (v + \lambda_0 v^3)B1(u, v)]/\Delta$;
6: else
7: $H1(0, 0) = \text{average depth}; H2(0, 0) = 0$;
8: end if

9: end for

Algorithm Part 3: Backward Transform

- 1: Calculate inverse Fourier transform of H1(u,v) and H2(u,v) in place: H1(x,y), H2(x,y);
- 2: for $(x, y) \in \Omega$ do
- 3: Z(x,y) = H1(x,y);
- 4: end for

Gradient Space

Example 1



Image triplet of a Beethoven statue used as input for 3PSM



40 20

Left: Recovered surface using the Frankot-Chellappa algorithm *Right*: Recovered surface using the Wei-Klette algorithm with $\lambda_0 = 0.5$ and $\lambda_1 = \lambda_2 = 0$

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Comments			

Constant $c_{\rm max}$ eliminates gradient estimates which define angles with the image plane close to $\pi/2$

A value such as $c_{max} = 12$ is an option

Real parts are stored in arrays A1, B1, and H1, and imaginary parts in arrays A2, B2, and H2

Average height can be estimated for the visible scene

Parameters λ_0 , λ_1 and λ_2 should be chosen based on experimental evidence for the given scene

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