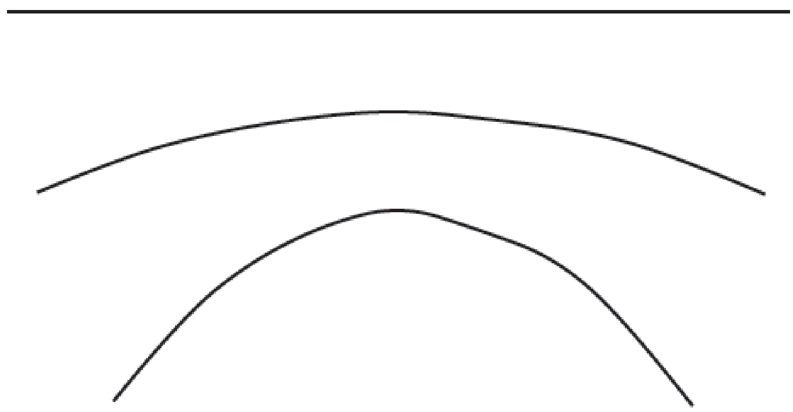
Curvature



curvature of a straight segment is zero

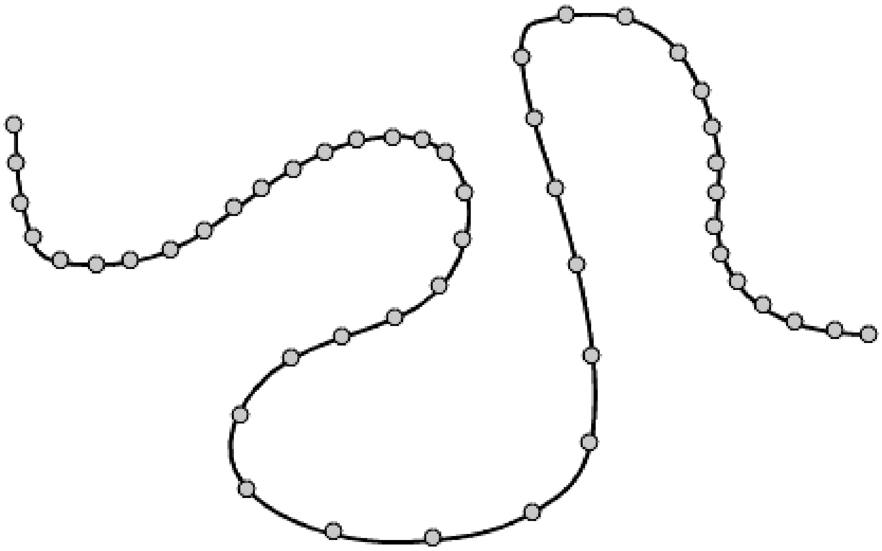
"more bending " = larger curvature

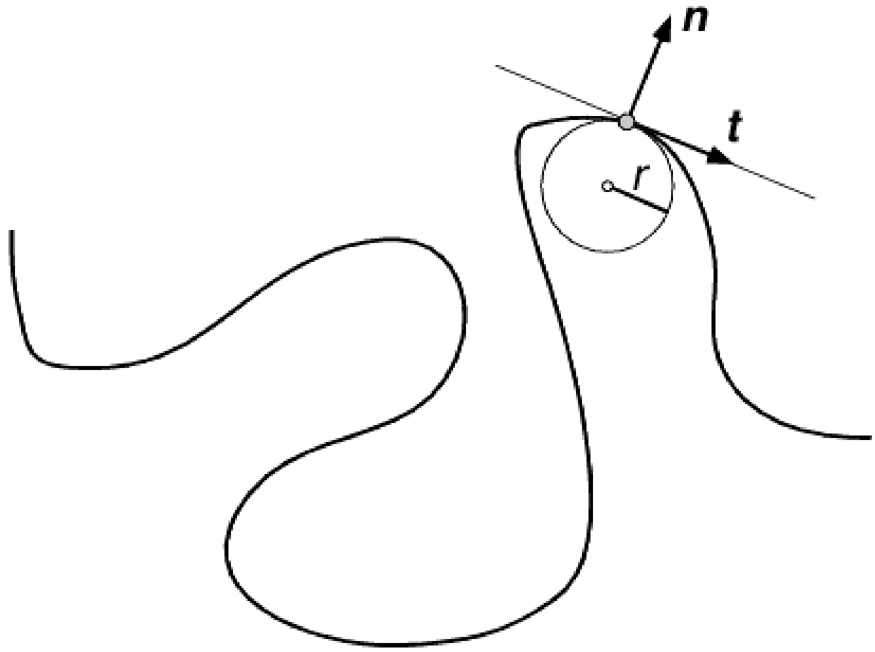
Corners

"corner" defined by large curvature value (e.g., a local maxima)

borders (i.e., edges in gray-level pictures) can be segmented into arcs at corners

A parameterized arc or curve γ is given at positions $\gamma(t)$ (see Lecture 05). Different densities of samples correspond to the *speed* of the parametrization.





Curvature at these positions $\gamma(t)$ can be characterized by *unit tangent vector* $\mathbf{t}(t)$, *unit normal vector* $\mathbf{n}(t)$, or the radius $r(t)$ of the *osculating circle*. Vectors $\mathbf{t}(t)$ and $\mathbf{n}(t)$ define the *Frenet frame* at $\gamma(t)$.

In vector notation, we can write

$$\gamma(t) = (x(t), y(t)) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2$$

where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ are the basis vectors of the Cartesian xy coordinate system. The tangent is defined by the first derivative

$$\dot{\gamma}(t) = (\dot{x}, \dot{y}) = (\dot{x}(t), \dot{y}(t)) = (\frac{\mathrm{d}x(t)}{\mathrm{d}t}, \frac{\mathrm{d}y(t)}{\mathrm{d}t})$$

of γ, assuming it is *smooth* (i.e., continuously differentiable). The unit tangent vector is as follows:

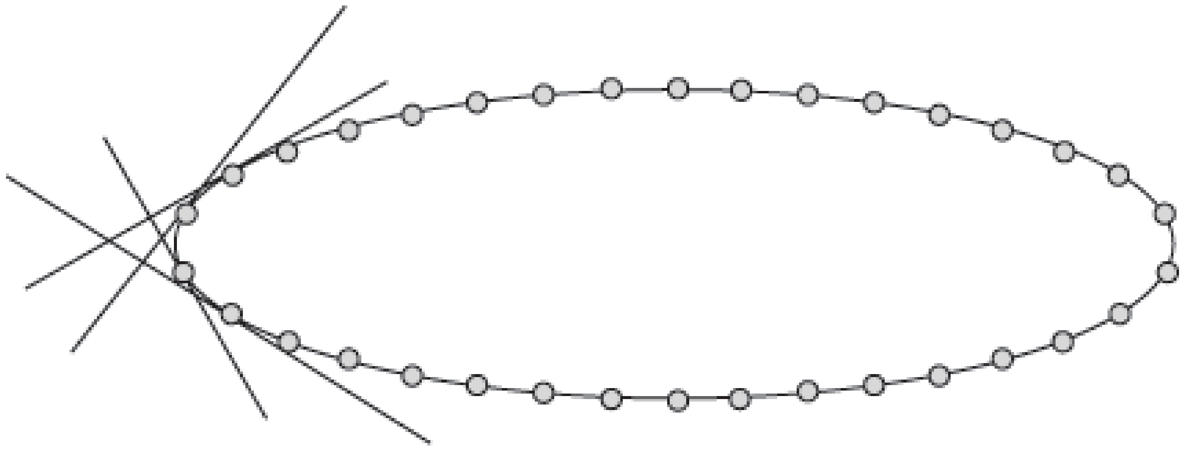
$$\mathbf{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$$

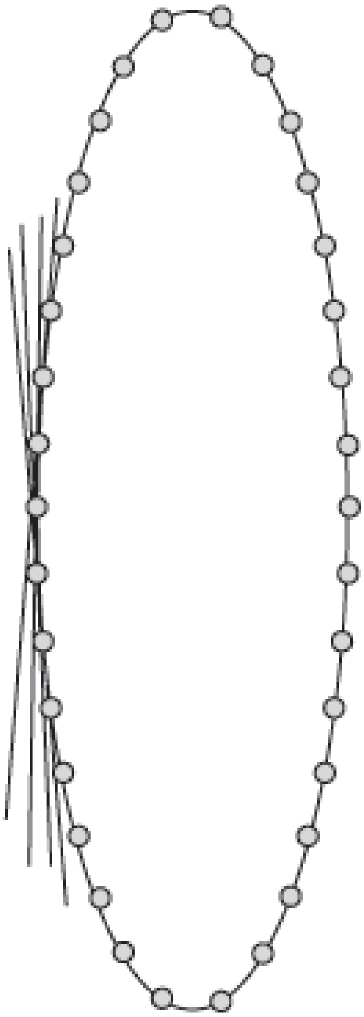
Vector $\mathbf{n}(t)$ is orthogonal to $\mathbf{t}(t)$, and both are assumed to form a right-hand coordinate system.

Let $l = \mathcal{L}(t)$ be the arc length between the starting point $\gamma(a)$ and the general point $p = \gamma(t)$. Curvature definition has to be independent of the *speed* (or rate of evolution)

$$v(t) = \frac{\mathrm{d}\mathcal{L}(t)}{\mathrm{d}t}$$

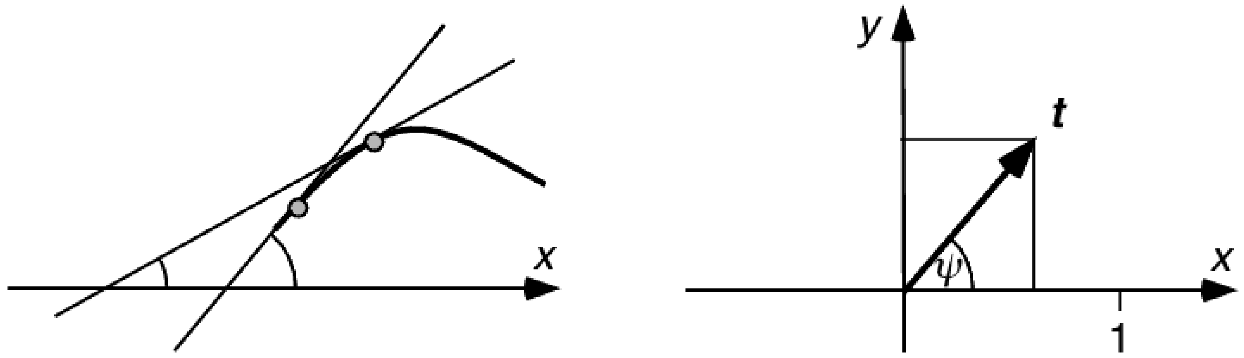
of the parameterization of γ.





Option 1: Rate of Turn

Curvature can be defined based on the rate of changes of the *slope angle* $\psi(t)$ between tangent $\mathbf{t}(t)$ and positive x-axis.



We have

$$\mathbf{t}(t) = (\cos\psi(t), \sin\psi(t))$$
$$= \cos\psi(t)\mathbf{e}_1 + \sin\psi(t)\mathbf{e}_2$$

and thus (note: orthogonal vector)

$$\mathbf{n}(t) = (-\sin\psi(t), \cos\psi(t))$$
$$= -\sin\psi(t)\mathbf{e}_1 + \cos\psi(t)\mathbf{e}_2$$

These formulae allow to calculate (estimate) $\psi(t)$, assuming we are able to calculate (estimate) either $\mathbf{t}(t)$ or $\mathbf{n}(t)$.

Definition 1 *The curvature of a smooth Jordan curve* γ *at* $\gamma(t) = (x(t), y(t))$ *is as follows:*

$$\kappa(t) = \frac{\mathrm{d}\psi(t)}{\mathrm{d}\mathcal{L}(t)} = \frac{\mathrm{d}\psi(t)}{\mathrm{d}l}$$

The application of this definition is supported by the following two

Frenet Formulae

$$\dot{\mathbf{t}}(t) = \kappa(t) \cdot v(t) \cdot \mathbf{n}(t)$$
$$\dot{\mathbf{n}}(t) = -\kappa(t) \cdot v(t) \cdot \mathbf{t}(t)$$

where

$$\begin{aligned} \dot{\mathbf{t}} &= (-\sin\psi \frac{\mathrm{d}\psi}{\mathrm{d}t}, \cos\psi \frac{\mathrm{d}\psi}{\mathrm{d}t}) \\ \dot{\mathbf{n}} &= (-\cos\psi \frac{\mathrm{d}\psi}{\mathrm{d}t}, -\sin\psi \frac{\mathrm{d}\psi}{\mathrm{d}t}) \end{aligned}$$

If we have parameterized curves (e.g., as preimages, before digitization, for evaluation studies in picture analysis) then we can use (one of) these Frenet formulae for calculating $\kappa(t)$.

If we use the arc-length parameterization (i.e., $t = l$) with unit speed $v(l) = 1$, the Frenet formulae simplify to the following:

$$\dot{\mathbf{t}}(l) = \kappa(l) \cdot \mathbf{n}(l)$$
$$\dot{\mathbf{n}}(l) = -\kappa(l) \cdot \mathbf{t}(l)$$

Example: A circle of radius r can be defined by the equation

$$x^2 + y^2 = r^2$$

or by the parameterization $\gamma(t) = (x(t), y(t))$, with

$$x = x(t) = r\cos t,\ \ y = y(t) = r\sin t$$

where $t \in [0, 2\pi)$.

For the arc length $\mathcal{L}(s)$ of a circle we obtain

$$\mathcal{L}(s) = \int_0^s \left(r^2 \sin^2 t + r^2 \cos^2 t\right)^{1/2} \mathrm{d}t = sr$$

(with $\mathcal{L}(2\pi) = 2\pi r$). This allows to calculate the speed of parametrization as

$$v(t) = r$$

Furthermore, starting with $\mathbf{t}(t) = \dot\gamma(t)/|\dot\gamma(t)|$ we obtain that

$$\mathbf{n} = \dot{\mathbf{t}}$$

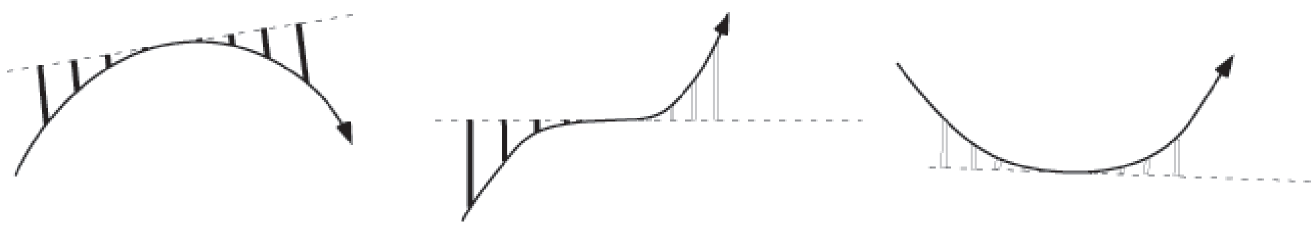
We have

$$\kappa(t) = 1/r$$

by the first Frenet formula.

Convex and Concave Points

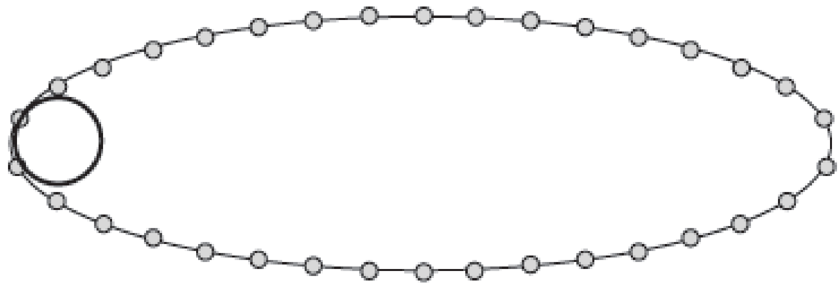
p is called a *convex point* of γ if the curvature of γ at p is positive and a *concave point* if it is negative.

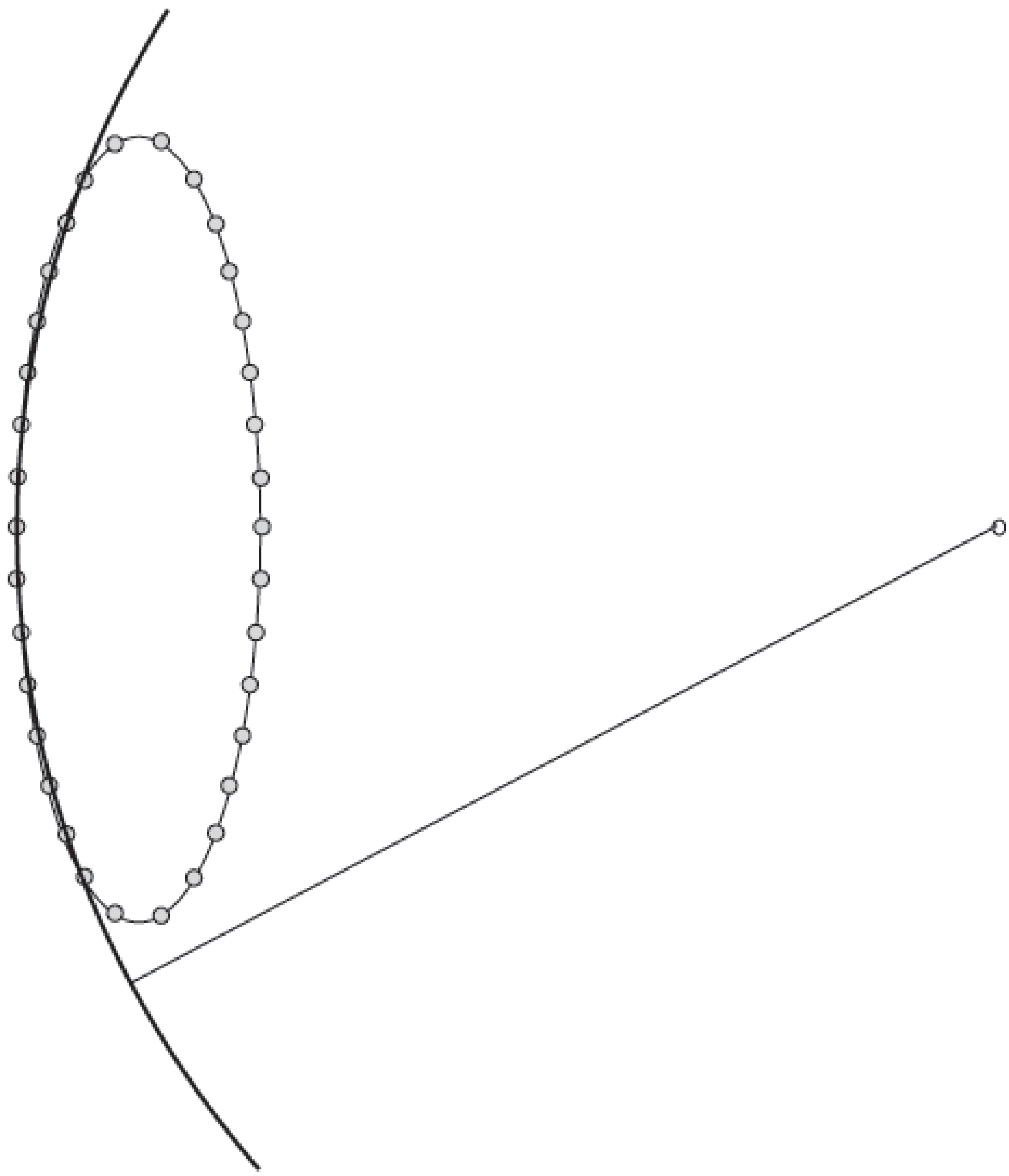


As the figure shows, the situation at p can be approximated by measuring the distances between γ and the tangent to γ at p along equidistant lines that are perpendicular to the tangent.

In the figure, positive distances are represented by bold line segments and negative distances by "hollow" line segments.

The area between the curve and the tangent line can be approximated by summing these distances; it is positive on the left (convex point), negative on the right (concave point), and zero in the middle, where the positive and negative distances cancel.

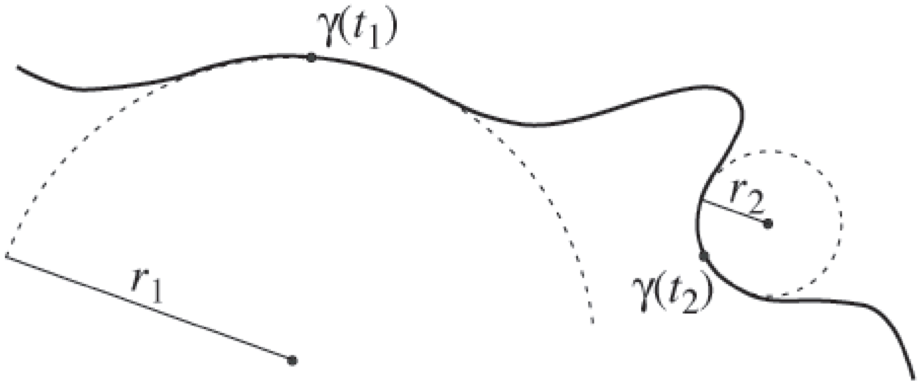




Option 2: Osculating Circle

The absolute value of the curvature at $p = \gamma(t)$ is equal to the inverse of the radius $r(t)$ of the *osculating circle* at p, which is the largest circle tangent to γ on the concave side of p.

The osculating circles at $p_i = \gamma(t_i)$, where $i = 1, 2$; the circle at p_i has radius $r(t_i)$:



Note that we cannot have $r(t) = 0$, but r is infinite when p is on a straight line segment.

In general, we have the following:

$$|\kappa(t)| = \frac{1}{r(t)} \quad \text{if } r(t) < \infty$$

The sign of κ depends on whether γ is convex or concave at $\gamma(t)$.

Conclusions

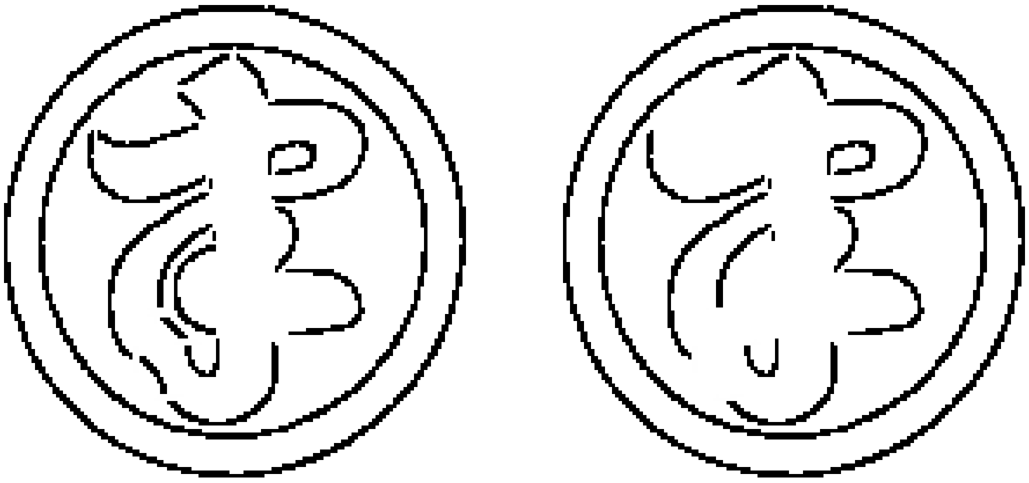
(1) The curvature of a digital curve can be estimated using either

Option 1 (i.e., approximations to the tangent vector, or, equivalently, estimated derivatives along the curve) or

Option 2 (i.e., approximations to the osculating circle).

Option 1 also allows to calculate the sign of curvature. See Lecture 22 for examples of methods.

(2) Classifications into convex or concave points are also useful for segmenting digital curves into convex or concave arcs.



Left: curves segmented at corners. Right: concave arcs are deleted.

Coursework

Related material in textbook: Section 8.1.3.

A.21. [5 marks] Page 2 of Lecture 20 specifies a set of test curves. Digitize the circle and the yin-part of the yinyang symbol for several (at least 20) different resolutions between $h = 32$ and $h = 1,024$. Segment the resulting borders (or frontiers, if you prefer the grid-cell model) in sequences of maximum-length DSSs (or 4-DSSs). A segment starts at a pixel p_i and ends at a pixel p_{i+k} (i.e., it approximates the DSS $p_i, p_{i+1}, \ldots, p_{i+k}$). Consider the middle point(s) $p_{i+k/2}$ (if k is even), or $p_{i+(k-1)/2}$ and $p_{i+(k+1)/2}$ (if k is odd) and the following *hypothesis*:

> The length $d_e(p_i, p_{i+k})$ of the DSS can be used as an approximate value of the radius of the osculating circle, and allows to estimate the curvature at this point (these points).

Your experiments (analysis of at least 40 digitized curves) should provide experimental evidence in favor or against this hypothesis.

Hint: DSS programs are available on the Internet (free downloads), in case that you have not already done Task **A.17**. Also, the curvature along the yin-curve is known because it consists of circular arcs.