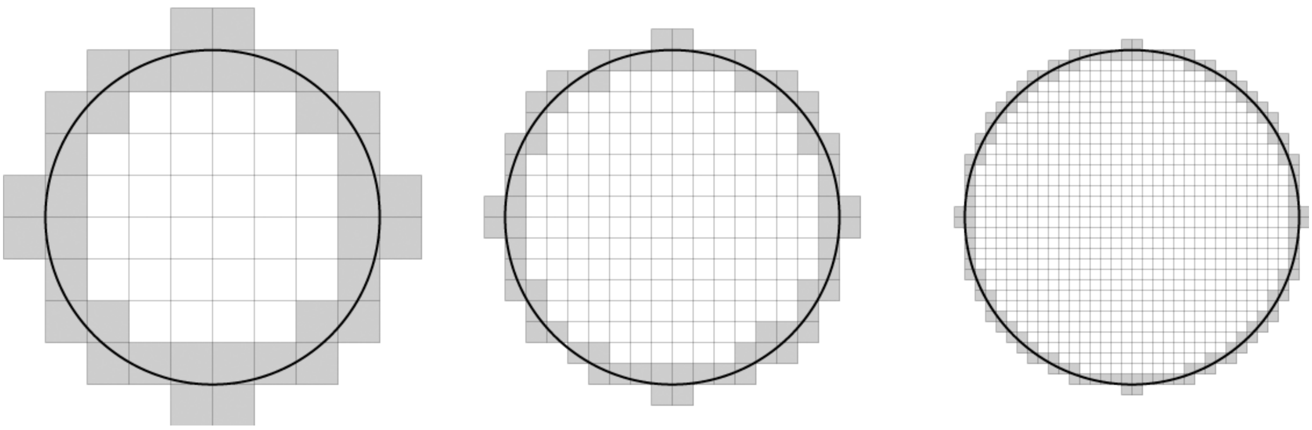
Jordan Digitization

introduced by C. Jordan in the late 19th century for measuring the contents of a set

Definition 1 *Let S be a nonempty closed subset of $\mathbb{R}^2$. Let $J_h^-(S)$ be the union of all 2-cells (for grid resolution $h > 0$) that are completely contained in the topological interior of S, and let $J_h^+(S)$ be the union of all such 2-cells that have nonempty intersections with S. $J_h^-(S)$ is called the* inner Jordan digitization of S *and $J_h^+(S)$ the* outer Jordan digitization *of S. For $S \subseteq \mathbb{R}^3$, we use 3-cells instead of 2-cells. For brevity, we denote J_1^- and J_1^+ with J^- and J^+, respectively.*



Inner and outer Jordan digitizations of a centered (i.e., midpoint at grid point) disk, for three different grid resolutions h

in general, the difference between outer and inner Jordan digitization can be transformed into a simple 1-curve by deletion of a few 2-cells, thus providing an ideal input for our MLP algorithm

Relations between Digitizations

assume a curve γ in the plane:

$$J_h^-(\gamma) = \emptyset \subseteq R_h(\gamma) \subseteq J_h^+(\gamma)$$

where R_h is the grid-intersection digitization in $\mathbb{Z}_h^2$ (set $J_h^+(\gamma)$ may contain additional pixels compared to $R_h(\gamma)$ if γ intersects grid edges at mid points)

If S is a nonempty proper subset of $\mathbb{R}^2$ or of $\mathbb{R}^3$ with a smooth frontier, we have $J_h^-(S) \subset J_h^+(S)$. Furthermore, the following is true:

$$J_h^-(S) \subseteq G_h(S) \subseteq J_h^+(S) \text{ for any } S \subseteq \mathbb{R}^2 \quad (S \subseteq \mathbb{R}^3)$$

One or both relations $\subseteq$ in the left part of this equation can be replaced by =, but both cannot if S has a smooth frontier.

Let S be a finite union of grid squares; then we have $J^-(S) = G(S) = J^+(S)$.

Types of Digital Sets

If S is, for example, a disk, square, or convex set (and similarly in 3D) we call $J_h^-(S), G_h(S)$, or $J_h^+(S)$ a *digital disk*, *digital square*, or *digital convex set*, respectively, provided it is connected.

We call a connected set of grid points a digital disk and so forth (with respect to a given digitization model), if there exists a disk and so forth that has that connected set as its digitization.

The topologic frontier of a simply connected compact set S in the Euclidean plane is a simple curve $\gamma : [0,1] \to \mathbb{R}^2$.

We assume that γ is rectifiable (i.e., has a defined length).

We are interested in estimating the length (later in the lecture: curvature) of γ from its digital representation.

$\mathbb{Z}_h^2$ and $\mathbb{Z}_h^3$ are grids (in the grid point model) with grid constant $0 < \theta \leq 1$ and grid resolution $h = 1/\theta$.

$\mathbb{Z}_h^2$ consists of grid points with coordinates that are $(\theta \cdot i, \theta \cdot j)$ where $i, j \in \mathbb{Z}$, and $\mathbb{Z}_h^3$ consists of grid points with coordinates that are $(\theta \cdot i, \theta \cdot j, \theta \cdot k)$ where $i, j, k \in \mathbb{Z}$.

Let $dig_h(\gamma)$ be a digitization of γ in $\mathbb{Z}_h^2$ (see definition of multigrid convergence).

Common 2D Curve Digitization Models

(i) a cyclic 8-path $\rho_{h,8}(\gamma)$ of grid points derived from grid-intersection digitization of γ in $\mathbb{Z}_h^2$;

(ii) a cyclic 4-path of vertices of 2-cells on the frontier of the Gauss digitization $G_h(S)$ of S; and

(iii) the closed difference set (in $\mathbb{R}^2$) between the outer and inner Jordan digitizations $A = J_h^+(S)$ and $B = J_h^-(S)$.

Method (iii) is applicable for length estimator E_{mlp}.

Local Length Estimation

historically the first ones in picture analysis, applied to digital curves defined by methods (i) or (ii); use of weighted distances in fixed neighborhoods; potentially of interest for short arcs or low-resolution curves; efficient linear online algorithms

the weights are chosen to approximate the Euclidean distance

(1) *best linear unbiased estimator* (BLUE) for length of γ (actually optimized for straight segments by L. Dorst an A.W.M. Smeulders, 1987):

$$E_{\mathrm{chm}}(\rho_{h,8}(\gamma)) = \frac{1}{h} \cdot (0{\cdot}948 \cdot n_i + 1{\cdot}343 \cdot n_d)$$

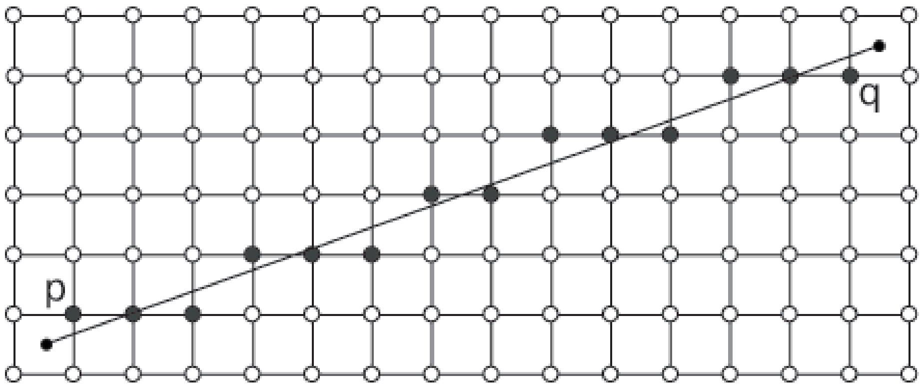
where n_i is the number of isothetic steps and n_d the number of diagonal steps in the digital arc or curve. (The subscript “chm” refers to the chessboard metric d_8.)

(2) *cornercount estimator* (coc), where n_c is the number of odd-even transitions in the chain code of the digital arc or curve (A.M. Vossepoel and A.W.M. Smeulders, 1982):

$$E_{\mathrm{coc}}(\rho_{h,8}(\gamma)) = \frac{1}{h} \cdot (0{\cdot}980 \cdot n_i + 1{\cdot}406 \cdot n_d - 0{\cdot}091 \cdot n_c)$$

Length of a Straight Segment

suppose a DSS pq has been detected; the distance $d_e(p,q)$ is typically slightly smaller than the original length of the segment γ prior to grid-intersection digitization



most probable original (mpo) length estimation method for DSS*s* (L. Dorst and A.W.M. Smeulders, 1991):

Let n be the length of the DSS $\rho_{h,8}(\gamma) = i(0), \ldots, i(n-1)$, and let a/b be the best possible rational estimate of its slope, where b is the length of the shortest period, which is the smallest $k \in \{1, \ldots, n\}$ such that $k = n$ or $i(m+k) = i(m)$ for $0 \leq m \leq n-k-1$. a is the height difference in one period; for example, if $i(m) \in \{0,1\}$ for all $0 \leq m \leq n-1$, then $a = i(0) + \ldots + i(q-1)$:

$$E_{\text{mpo}}(\rho_{h,8}(\gamma))) = \frac{1}{h} \cdot (n\sqrt{1+(a/b)^2})$$

In the figure: $\rho_{h,8}(\gamma) = 0010010100100$, $n = 13$, $b = 8$, and $a = 3$; we have $E_{\text{mpo}}(\rho_{h,8}(\gamma))) = 13.8840\ldots/h$, compared to $d_e(p,q) = 13.6015\ldots/h$

DSS Estimators

(1) basic DSS estimator: segment a digital arc or curve into a sequence of maximum-length DSS*s*; a length estimator E_{DSS} is then defined by the length of the resulting polygon or polygonal arc (the segmentation into DSS*s* depends on the method, the chosen starting point, and the direction in which the arc or curve is traced)

(1.a) algorithm **DR1995** (for 8-curves) defines the $E_{8\mathrm{ss}}$ estimator

(1.a) algorithm **K1990** (for 4-curves) defines the $E_{4\mathrm{ss}}$ estimator

(2) refined DSS estimator: instead of using the length of the polygonal curve, consider each edge of the polygonal curve as a digitization of a straight segment, and use a refined length estimation for such segments.

(2.a) algorithm **DR1995** (for 8-curves) and sum the mpo length estimates of the 8-DSS*s*; this defines the $E_{8\mathrm{mp}}$ estimator

Tangent Based Estimators

assume a curve $\gamma(t) = (x(t), y(t))$ in the plane, with $a \leq t \leq b$

speed of this parameterization:

$$v(t) = \|\dot{\gamma}(t)\|_2 = \|(\dot{x}, \dot{y})\|_2 = \|(\dot{x}(t), \dot{y}(t))\|_2$$

where

$$\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \text{ and } \dot{y}(t) = \frac{\mathrm{d}y(t)}{\mathrm{d}t}$$

Example: a circle is given by $x = x(t) = r\cos t$ and $y = y(t) = r\sin t$, for $t \in [0, 2\pi)$, and

$$v(t) = \|(-r\sin t, r\cos t)\|_2 \ = \ r\sqrt{\sin^2 t + \cos^2 t} \ = \ r$$

is the speed of this parametrization

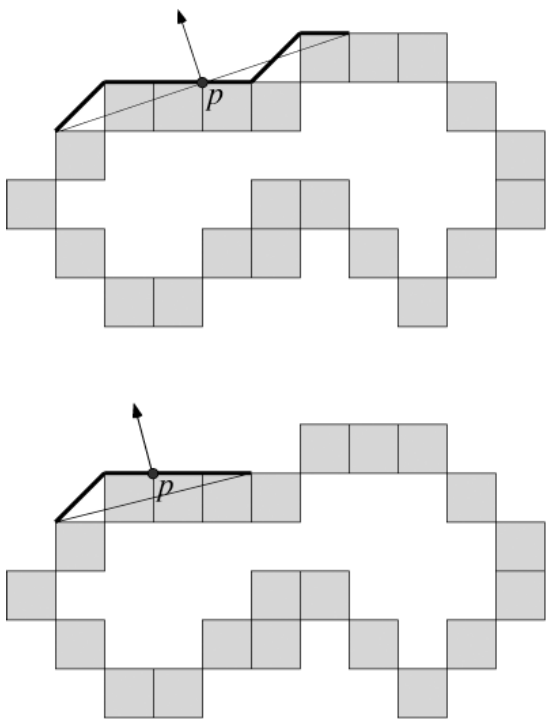
In Lecture 05 we considered a parametrizable Jordan curve or arc γ, and defined its length $\mathcal{L} = \|\gamma\|_2$ as follows, where $a \leq t \leq b$:

$$\mathcal{L}(t) = \int_a^t \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}s = \int_a^t v(s)\,\mathrm{d}s$$

Let $\|(\dot{x}, \dot{y})\| : [a, b] \to \mathbb{R}^2$ be the length of the *tangent vector* associated with $\gamma(t)$. Then the following can be approximated by using discrete estimates of the products $\|(\dot{x}, \dot{y})\| \ \mathrm{d}t$:

$$\mathcal{L}(\gamma) = \int_a^b \|(\dot{x}, \dot{y})\| \ \mathrm{d}t$$

The DSS (or 4-DSS) approximates the tangent line to γ, and the *normal* is perpendicular to this line. For example, we consider frontier tracing, and a 0-cell p as the center point of a maximum-length DSS (or 4-DSS).



Left column: A DSS centered at p. Right column: A 4-DSS centered at p. Upper row: estimation of $\mathbf{n}_1$; both estimated tangential lines — and hence both estimated normals — are identical. Lower row: estimation of $\mathbf{n}_2$; different estimated tangential lines result in different estimated normals, depending on how the estimator is defined.

Example: let S be the set of all 1-cells in the alternating sequence of 0-cells and 1-cells along the chosen frontier, calculate normal $\mathbf{n}(c)$ and local increment $\mathbf{n}_0(c)$ based on DSSs; the length estimator is

$$E_{\tan}(dig_h(\gamma)) = \sum_{c \in S} \mathbf{n}(c) \cdot \mathbf{n}_0(c)$$

Convergence of Length Estimators

We apply the general definition of multigrid convergence (Lecture 06) to the problem of estimating the length $\mathcal{L}(\gamma)$ of an arc or curve γ.

Assume that the estimate E is defined for all curves γ in a given class (e.g., the class of all simple curves in the Euclidean plane) and for all digitizations $dig_h(\gamma)$ where $h > 0$. E is multigrid convergent to $\mathcal{L}$ with respect to digitization model dig_h iff $E(dig_h(\gamma))$ converges to $\mathcal{L}(\gamma)$ as $h \to \infty$ for any curve γ in the class of interest.

More formally, we have the following:

$$|E(dig_h(\gamma)) - \mathcal{L}(\gamma)| \leq \kappa(h)$$

where $\lim_{h\to\infty} \kappa(h) = 0$; the speed of convergence is $\mathcal{O}(1/\kappa(h))$.

Example

estimator: E_{mpo}

class of curves: straight segments

digitization: grid-intersection digitization

result: superlinear convergence with κ in $O(h^{-1 \cdot 5})$

Multigrid Convergence Theorems

The local estimators chm and coc of the lengths of digitized arcs or curves are not multigrid convergent (also indicated by experiments reported in next lecture).

Given an algorithm for constructing a DSS approximation γ_h of the *h-frontier* $\vartheta_h(S)$ of a simply connected digital set $G_h(S)$, we define $\varepsilon_{\mathrm{DSS}}(h) = \varepsilon_{\mathrm{DSS}}/h$ as the maximum Hausdorff distance between $\vartheta_h(S)$ and γ_h:

$$d_e(\vartheta_h(S), \gamma_h) \leq \frac{\varepsilon_{\mathrm{DSS}}}{h}$$

Theorem 1 (R. Klette and J. Zunic, 2000) *Let S be a convex h-compact polygonal set in $\mathbb{R}^2$. Then there exists a grid resolution h_0 such that, for all $h \geq h_0$, any DSS approximation γ_h of the h-frontier $\vartheta_h(S)$ is a connected polygon with perimeter $\mathcal{P}_h$ that satisfies the following inequality:*

$$|\mathcal{L}(\vartheta(S)) - \mathcal{P}_h| \leq \frac{2\pi}{h}\left(\varepsilon_{\mathrm{DSS}}(h) + \frac{1}{\sqrt{2}}\right)$$

(S is called *h-compact* iff there is an $h_0 > 0$ such that $\vartheta_h(S)$ is a single (connected) curve for any $h \geq h_0$.)

Example: For $\varepsilon_{\mathrm{DSS}} = 1$ (as, e.g., in case of algorithm **DR1995**) we have $\varepsilon_{\mathrm{DSS}}(h) = 1/h$, and we obtain

$$\frac{2\pi}{h^2} + \frac{2\pi}{h \cdot \sqrt{2}} \approx \frac{\pi\sqrt{2}}{h} \approx \frac{4{\cdot}5}{h} \quad \text{if } h \text{ is large}$$

(i.e., linear convergence speed for any [!] constant $\varepsilon_{\mathrm{DSS}}$).

Theorem 2 (F. Sloboda, B. Zatko, and J. Stoer, 1998) *Let γ be a convex planar curve that is contained in a simple 1-curve ρ in the plane for grid resolution $h \geq 1$. Then the MLP approximation of ρ is a connected polygonal curve of length $\mathcal{P}_h$ that satisfies the following:*

$$\mathcal{P}_h \leq \mathcal{L}(\gamma) < \mathcal{P}_h + \frac{8}{h}$$

In other words, the convergence speed is upper bounded by $8/h$ in this case. The determination of optimum error bounds (for DSS and mlp) remains an open problem.

Theorem 3 (D. Coeurjolly and O. Teytaud, 2001) *Let γ be a simple $C^{(2)}$ curve with bounded curvature then both the estimated discrete tangent direction and the tangent-based length estimate $E_{\tan}(dig_h(\gamma))$ are multigrid convergent.*

The speed of convergence and the maximum error bound for these estimates have not yet been determined.

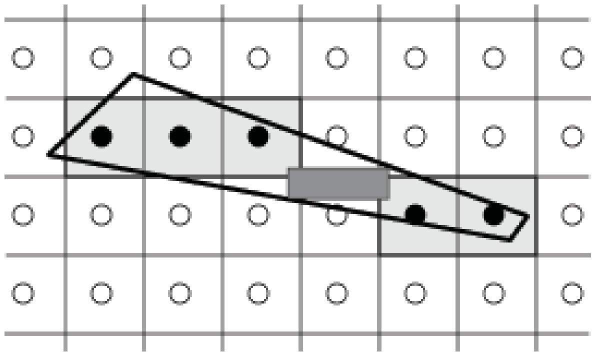
Coursework

Related material in textbook: Sections 2.3.2, 2.3.4, 8.1.1, 10.1, 10.2.1, 10.2.2, 10.2.3, 10.2.6, and 10.3.2. Solve Exercise 2 on page 372.

A.19. [5 marks] Do Exercise 1 on page 372. (Note the similarity with Exercise 6.2 in these lecture notes, where the convex hull has been used for perimeter estimation). A solution to this exercise requires that you already have implementations for DSS or MLP length estimations at hand. However, note that these can be downloaded (for free) from the net.

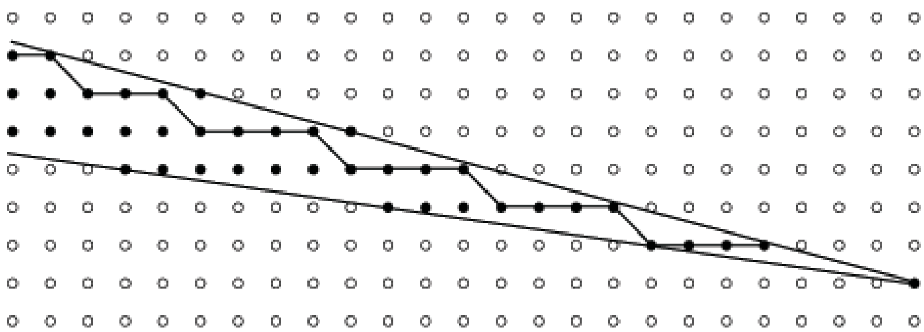
Appendix: Proof of Theorem 1

Why do we request connectedness and h-compactness?



A simple convex polygon S for which $G_h(S)$ splits into two components (dark shaded rectangle: area where components can be reconnected at a higher grid resolution).

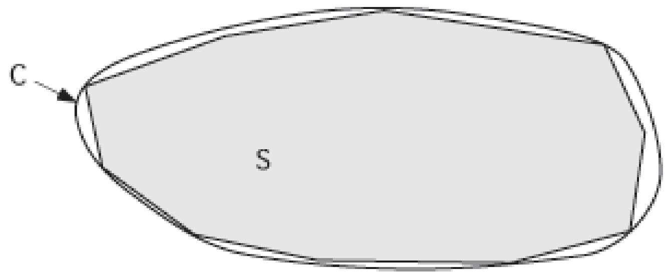
Situations in which components cannot be connected for any grid resolution can arise at a vertex with a small interior angle:



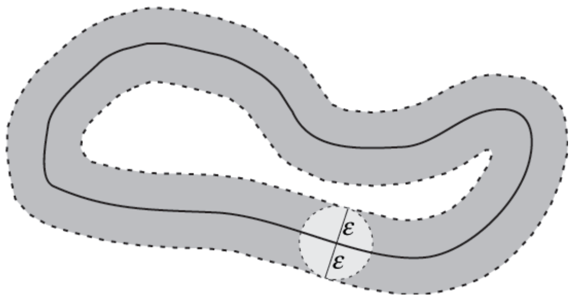
There is a ray with rational slope $1/5$ "inside" the shown sector; any grid resolution $5nh$ ($n \geq 1$) leads to the same situation.

We use two lemmas from integral geometry:

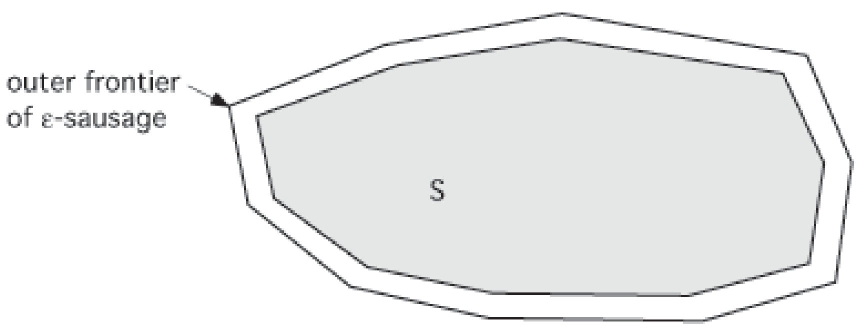
Lemma 1 *If a convex planar polygonal set S is contained in a convex planar set C, the perimeter $\mathcal{P}(S)$ of S is at most equal to $\mathcal{P}(C)$.*



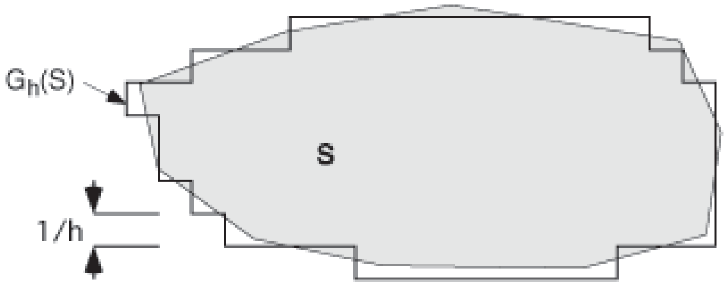
The *ε-sausage* of a curve γ (H. Minkowski, 1910) is the set of all points p such that $d_e(\{p\}, \gamma) \leq \varepsilon$.



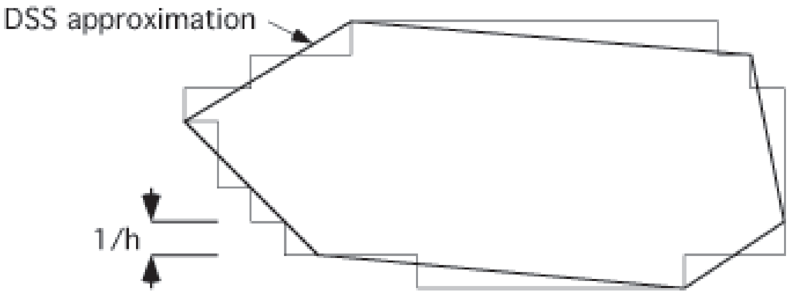
Lemma 2 *The length of the outer frontier of the ε-sausage of the frontier of a convex planar polygon S is $\mathcal{P}(S) + 2\pi\varepsilon$.*



We now prove Theorem 1.

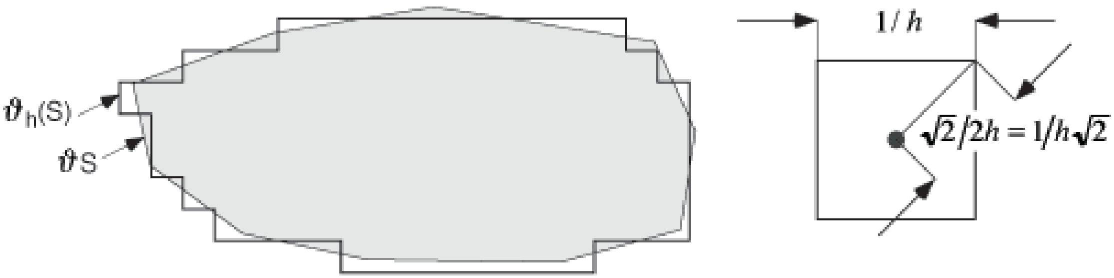


S is h-compact for $h \geq h_0$, so $G_h(S)$ is connected, and any DSS approximation of $\vartheta_h(S)$ is a single (connected) polygonal curve.



(Proposition 1) At first we prove that there exists a constant $h_1 \geq 1$ such that the following is true for all $h \geq h_1$:

$$d_e(\vartheta S, \vartheta_h(S)) \leq \frac{1}{h \cdot \sqrt{2}} \tag{1}$$



Suppose this Hausdorff distance was greater than $(h \cdot \sqrt{2})^{-1}$. Then there would exist either

(A) at least one point p on $\vartheta_h(S)$ with a minimum Euclidean distance to ϑS that is greater than $(h \cdot \sqrt{2})^{-1}$ or

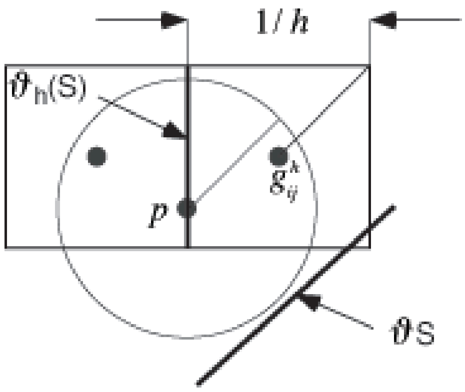
(B) at least one point q on ϑS with a minimum Euclidean distance to $\vartheta_h(S)$ that is greater than $(h \cdot \sqrt{2})^{-1}$.

(A): The circle with center p and radius $(h \cdot \sqrt{2})^{-1}$ would not contain any point of ϑS; hence this circle would be either

(A1) disjoint from S or

(A2) completely inside of S.

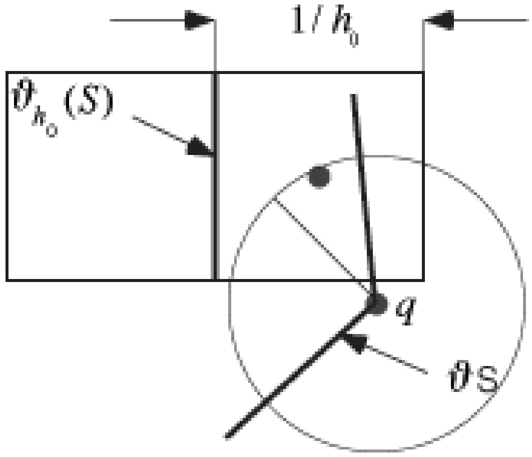
Let p be on the frontier of a grid square with midpoint g_{ij}^h. The grid point g_{ij}^h is inside the circle.



In case **(A1)**, it follows that g_{ij}^h cannot be in S (i.e., g_{ij}^h is not in $G_h(S)$). It follows that p cannot be on $\vartheta_h(S)$, which contradicts our assumption.

In case **(A2)**, p is on an h-edge incident with two h-squares with midpoints that are both in the circle and thus in S; hence, in this case, too, p cannot be on $\vartheta_h(S)$. Thus case **(A)** is impossible.

(B): Because S is h-compact for $h \geq h_0$, the distance between q and the nearest grid point can become arbitrarily small as $h \to \infty$, while $G_h(S)$ still remains connected. Thus, in case **(B)**, we must increase h so that $h \geq h_1 \geq h_0$ for some h_1, which represents the situation in which the minimum Euclidean distance from $q \in \vartheta S$ to $\vartheta_h(S)$ is less than or equal to $(h \cdot \sqrt{2})^{-1}$.



Only a finite number of vertices on the frontier of the polygonal set S needs to be considered for such increases in h_0. This concludes the proof of Proposition 1.

(Proposition 2) The defined upper bound

$$d_e(\vartheta_h(S), \gamma_h) \leq \frac{\varepsilon_{\mathrm{DSS}}}{h}$$

and Equation (1) and the triangle inequality for Hausdorff distance imply the following:

$$d_e(\vartheta S, \gamma_h) \leq d_e(\vartheta_h(S), \gamma_h) + d_e(\vartheta S, \vartheta_h(S)) \leq \frac{\varepsilon_{\mathrm{DSS}}}{h} + \frac{1}{h \cdot \sqrt{2}}$$

(Proposition 3) Let $\varepsilon = \varepsilon_{\mathrm{DSS}}/h \ + \ 1/(h \cdot \sqrt{2})$. Then the perimeter of S and the length of γ_h differ by at most $2\pi\varepsilon$. (See V. Kovalevsky and S. Fuchs, 1992)

Let the constructed DSS approximation γ_h be the frontier of a convex (polygonal) set C so that (see Proposition 2):

$$d_e(\vartheta S, \vartheta C) \leq \varepsilon \qquad (2)$$

Note that $\vartheta C = \gamma_h$. Proposition 3 says that the following is true:

$$|\mathcal{P}(S) - \mathcal{P}(C)| \leq 2\pi\varepsilon \qquad (3)$$

Note that $\mathcal{P}(S) = \mathcal{L}(\vartheta(S))$ and $\mathcal{P}(C) = \mathcal{L}(\gamma_h) = \mathcal{P}_h$. So we have

$$|\mathcal{L}(\vartheta(S)) - \mathcal{P}_h| \leq 2\pi\varepsilon = 2\pi \left(\varepsilon_{\mathrm{DSS}}/h \ + \ 1/(h \cdot \sqrt{2})\right)$$

and this is the statement in Theorem 1.

What remains is that we have to prove Proposition 3.

The frontier ϑC lies in the ε-sausage of the frontier ϑS. Let $\vartheta_\varepsilon S$ be the outer frontier of the ε-sausage of ϑS. By Lemma 1, we have the following:

$$\mathcal{P}(C) \leq \mathcal{L}(\vartheta_\varepsilon S)$$

By Lemma 2, we have the following:

$$\mathcal{L}(\vartheta_\varepsilon S) = \mathcal{P}(S) + 2\pi\varepsilon$$

Hence, the following is true:

$$\mathcal{P}(C) \leq \mathcal{P}(S) + 2\pi\varepsilon \tag{4}$$

ϑS lies in the ε-sausage of ϑC, because Hausdorff distance is symmetric. Let $\vartheta_\varepsilon C$ be the outer frontier of the ε-sausage of ϑC. By Lemma 1, we have the following:

$$\mathcal{P}(S) \leq \mathcal{L}(\vartheta_\varepsilon C)$$

By Lemma 2, we have the following:

$$\mathcal{L}(\vartheta_\varepsilon C) = \mathcal{P}(C) + 2\pi\varepsilon$$

Hence, the following is true:

$$\mathcal{P}(S) - 2\pi\varepsilon \leq \mathcal{P}(C) \tag{5}$$

From Equations 4 and 5, we have the following,

$$\mathcal{P}(S) - 2\pi\varepsilon \leq \mathcal{P}(C) \leq \mathcal{P}(S) + 2\pi\varepsilon$$

which proves Proposition 3, and thus Theorem 1.

Q.E.D.