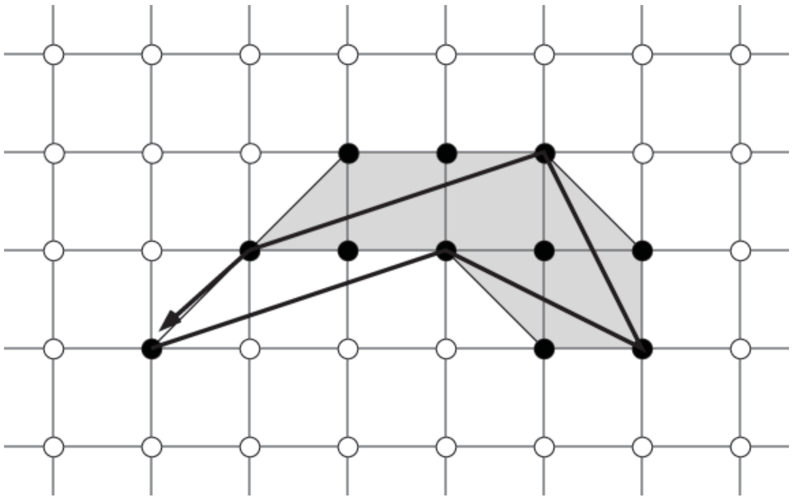
A Linear Online DSS Algorithm

I. Debled-Rennesson 1995: algorithm **DR1995**

We consider the recognition of (8-)DSS*s* on the border of a region in the 2D (picture) grid.



Algorithm **DR1995** is an efficient linear online DSS recognition algorithm; it has been also applied for DSS recognition in 3D (see textbook, Section 11.1).

It is based on the calculation of a *narrowest strip* defined by the nearest support below and above (see Theorem 1 on page 4 in Lecture 15). The mathematical background for this approach is *arithmetic geometry* (established by J.-P. Reveillès in 1991; further developed by, e.g., E. Andres, R. Acharya, and C. Sibata in 1997).

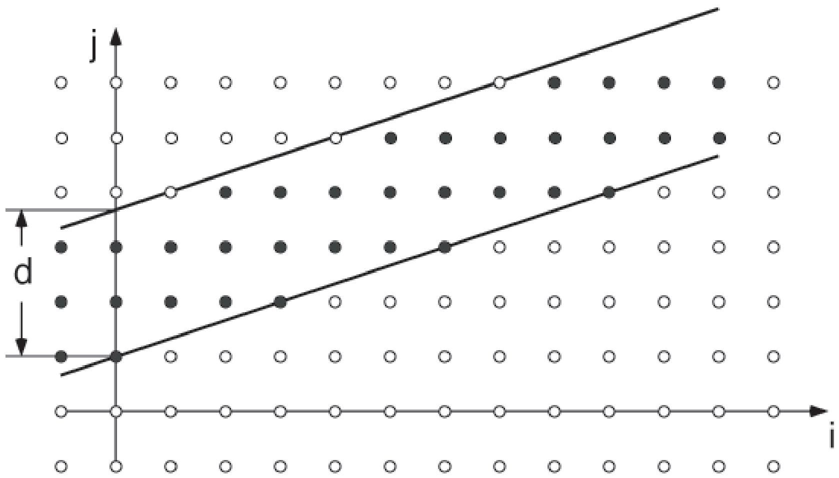
Arithmetic Geometry

DSS*s* (or digital planes) are defined in arithmetic geometry by pairs of linear Diophantine (i.e., only integer parameters) inequalities.

In the 2D case, let a and b be relatively prime integers, let c and w be integers:

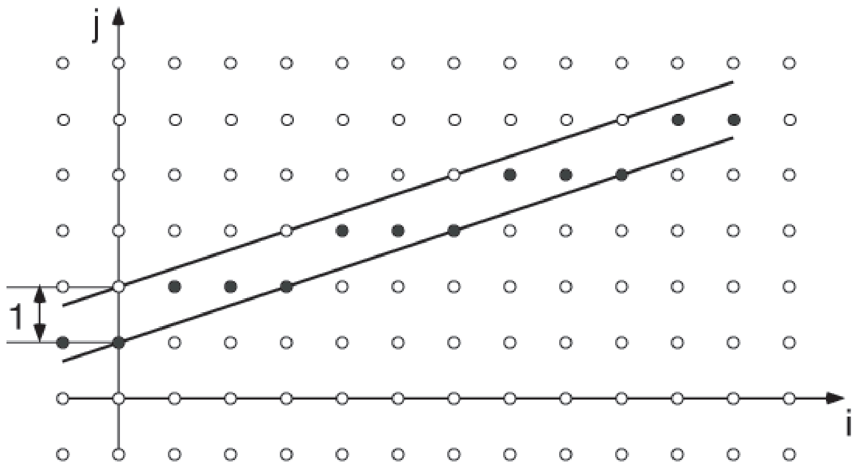
$$D_{a,b,c,w} = \{(i,j) \in \mathbb{Z}^2 : c \leq bi - aj < c + w\}$$

$D_{a,b,c,w}$ is called an *arithmetic line* with *slope* b/a, *approximate intercept* c/a, and *arithmetic width* $|\frac{w}{a}|$.



The two supporting straight lines are $j = \frac{b}{a}i - \frac{c}{a}$ and $j = \frac{b}{a}i - \frac{c}{a} - \frac{w}{a}$. It is $d = |\frac{w}{a}|$. (The actual (i.e., geometric) intercept could be specified by $-\frac{3c+w}{2a}$.)

Digital Straight Lines



Theorem 1 *Any set of grid points* $D_{a,b,c,\max\{|a|,|b|\}}$ *is the set of grid points of a digital straight line. Conversely, for any rational digital straight line, there exist* a, b, *and* c *such that the set of grid points of the given digital straight line is* $D_{a,b,c,\max\{|a|,|b|\}}$.

This theorem states that $w = \max\{|a|, |b|\}$ defines digital straight lines, and as subsets also digital 8-rays or 8-DSS*s*.

$w = |a| + |b|$ defines (compare the characterization of 4-DSS*s* in Lecture 16) digital 4-rays or 4-DSS*s*.

Specification for Algorithm DR1995

Assumption (w.l.o.g.):

We are considering lines with slope $0 \leq b/a \leq 1$; thus we have $0 \leq b \leq a$. This allows in the following that we can use a instead of $\max\{|a|, |b|\}$.

Resulting specifications for first octant:

For a naive line, we have $w = a$, and all the grid points in $D_{a,b,c,w}$ lie between or on two lines $bx - ay = c$ and $bx - ay = c + a - 1$.

These are the two supporting lines of $D_{a,b,c,a}$.

Test in the algorithm:

Grid points (x, y) on an 8-DSS must satisfy the constraint

$$D: \ c \leq bx - ay < c + a$$

where a and b are relatively prime integers and c is an integer. The algorithm has to provide values of a, b, and c.

Algorithm DR1995

Initialization for a new segment

The initial grid point q_1 of a new 8-DSS is identified with the origin $(0, 0)$.

We assume that the given sequence of grid points involves moves in at most two directions: $(1, 0)$ or $(1, 1)$; thus we proceed in the x-direction in the first octant.

Other point sequences are mapped into this by reflection.

At $q_1 = (0, 0)$, we start with condition

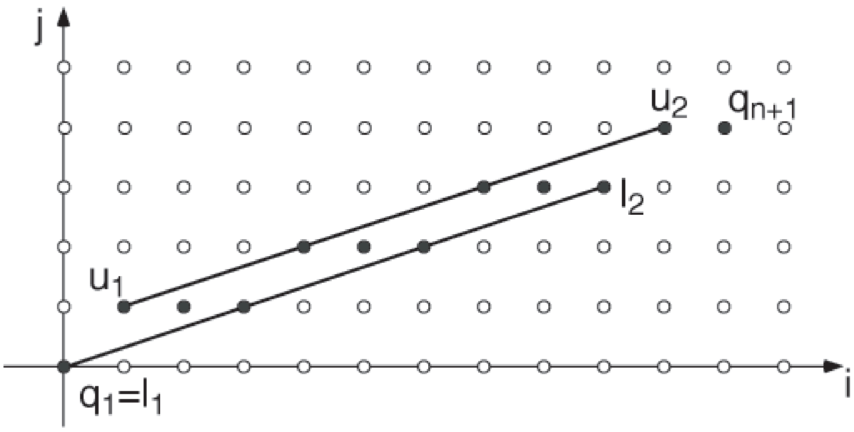
$$D_1 : \ 0 \leq -y < 1$$

We have $a = 1$, $b = 0$, and $c = 0$.

Algorithm DR1995

Test and updating for next grid point

For a new point $q_{n+1} \in A_8(q_n)$, where $x_{n+1} > x_n$, we have three cases: q_{n+1} is between or on these two lines (i.e., no update is needed) or it is ("just") above the upper or ("just") below the lower supporting line.



Let u_1, u_2 and l_1, l_2 be the points on the upper and lower supporting line, respectively, where index 1 denotes the point q_i $(1 \leq i \leq n)$ with the smallest x-coordinate and index 2 denotes the point with the largest x-coordinate (compare points *StartN*, *EndN* and *StartP*, *EndP* in Lecture 16).

Basic Theorem for Algorithm

Slope b/a is as calculated so far for the initial 8-DSS $\{q_1, \ldots, q_n\} \subset D_{a,b,c,a}$.

The *remainder* of point q_{n+1} with respect to slope b/a is

$$bx_{n+1} - ay_{n+1}$$

Let $r = bx_{n+1} - ay_{n+1} - c$.

Theorem 2 (CASE 0) *If* $0 \leq r < \max\{|a|, |b|\}$, *then* $q_{n+1} \in D_{a,b,c,\max\{|a|,|b|\}}$.

(CASE 1) *If* $r = -1$, *then* $\{q_1, \ldots, q_n, q_{n+1}\}$ *is an 8-DSS with a slope that is defined by vector* $u_1 q_{n+1}$.

(CASE 2) *If* $r = \max\{|a|, |b|\}$, *then* $\{q_1, \ldots, q_n, q_{n+1}\}$ *is an 8-DSS with a slope that is defined by vector* $l_1 q_{n+1}$.

(CASE 3) *If* $r < -1$ *or* $r > \max\{|a|, |b|\}$, *then* $\{q_1, \ldots, q_n, q_{n+1}\}$ *is not an 8-DSS.*

Algorithm DR1995

(for the first octant; $\max\{|a|, |b|\} = a$)

1. Let $r = bx_{n+1} - ay_{n+1} - c$ be the remainder of the new point q_{n+1} minus c.
2. If $0 \leq r < a$, then $u_2 = q_{n+1}$ (if $r = 0$) or $l_2 = q_{n+1}$ (if $r = a - 1$), and stop; otherwise, go to Step 3.
3. If $r = -1$, then

   $$l_1 = l_2, u_2 = q_{n+1}, a = |x_{n+1} - u_{11}|, b = |y_{n+1} - u_{12}|$$

   (where $u_1 = (u_{11}, u_{12})$),

   $$c = bx_{n+1} - ay_{n+1}$$

   and stop; otherwise, go to Step 4.
4. If $r = a$, then

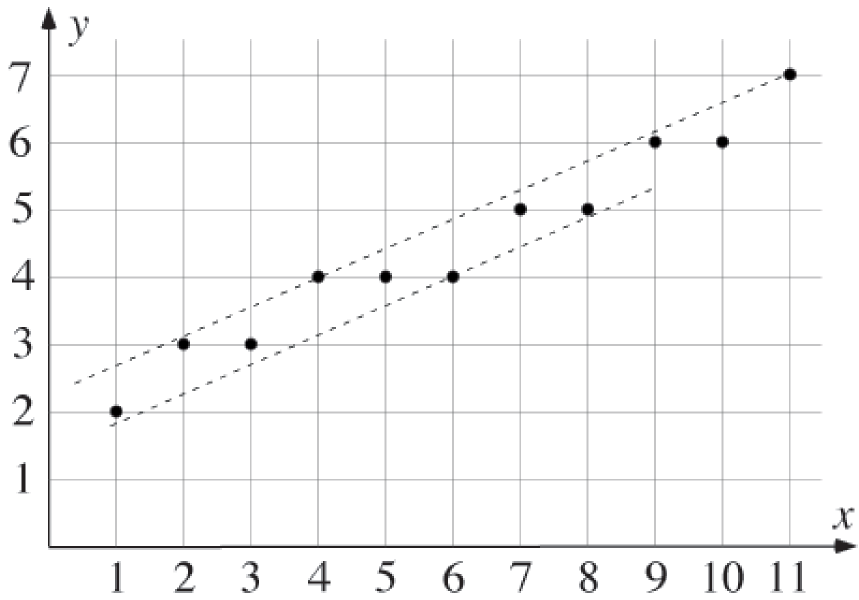
   $$u_1 = u_2, l_2 = q_{n+1}, a = |x_{n+1} - l_{11}|, b = |y_{n+1} - l_{12}|$$

   (where $l_1 = (l_{11}, l_{12})$),

   $$c = bx_{n+1} - ay_{n+1} - a + 1 \text{ (or } c = bu_{21} - au_{22})$$

   and stop; otherwise, go to Step 5.
5. The new point does not form a DSS with the previous n points; initialize a new DSS at q_n.

Example



This is a DSS with $p_1 = (1,2), p_2 = (2,3), \ldots, p_{11} = (11,7)$. We show that the algorithm accepts this 8-arc.

We consider these points as a sequence $q_i = p_i - p_1$ in the first octant, with $q_1 = (0,0)$ and $i = 1, \ldots, 11$.

n=2: q_1 and q_2 satisfy condition $D_2 : \ 0 \leq x - y < 1$ with $c = 0$; the upper supporting lines are both $y = x$.

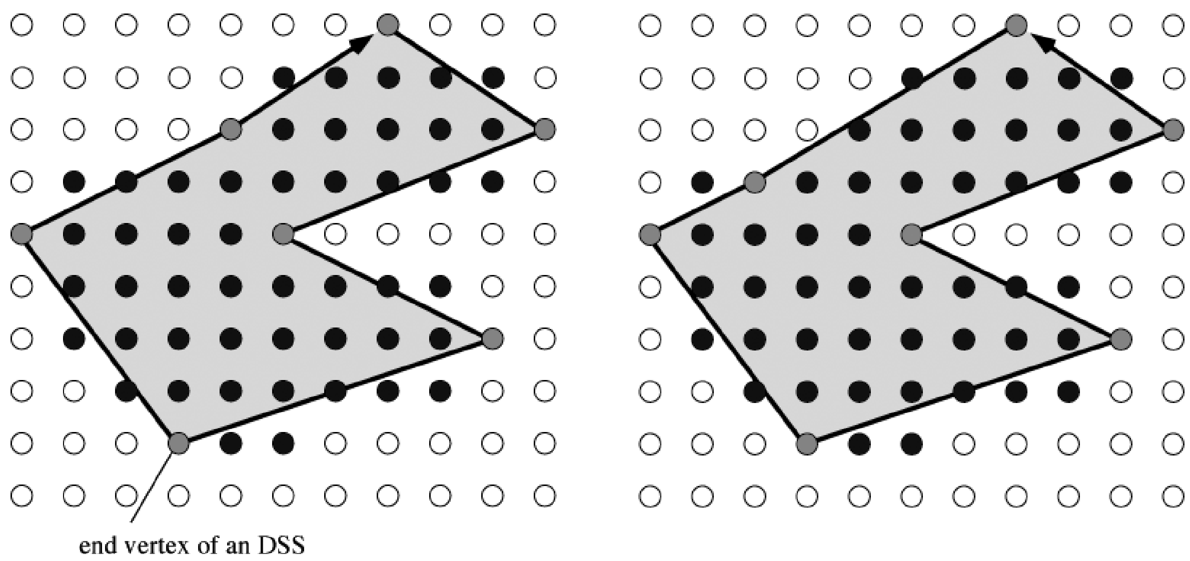
n=3: q_1, q_2, and q_3 satisfy condition $D_3 : \ -1 \leq x - 2y < 1$; here, the lower supporting line is $2y = x$ and $c = -1$ is defined by q_2.

n=6: At $q_6 = (5,2)$, we have condition $D_6 : \ -4 \leq 2x - 5y < 1$; $c = -4$, $a = 5$, and $b = 2$, and the upper supporting line is defined by q_4.

n=10: At q_{10}, we still have a DSS $q_1, \ldots, q_{10}$ with $l_1 = q_1$, $l_2 = q_6$, $u_1 = q_4$, and $u_2 = q_9$.

n=11: The remainder r at q_{11} is -5; we have (CASE 1) in Theorem 2. The new slope is defined by vector $u_1 q_{11} = (7, 3)$, and $\{q_1, \ldots, q_{11}\}$ is a segment of the digital line $D_{11}: \ -5 \leq 3x - 7y < 2$. We now have $l_1 = l_2 = q_6$, $u_1 = q_4$, and $u_2 = q_{11}$.

Non-Unique Segmentation Results



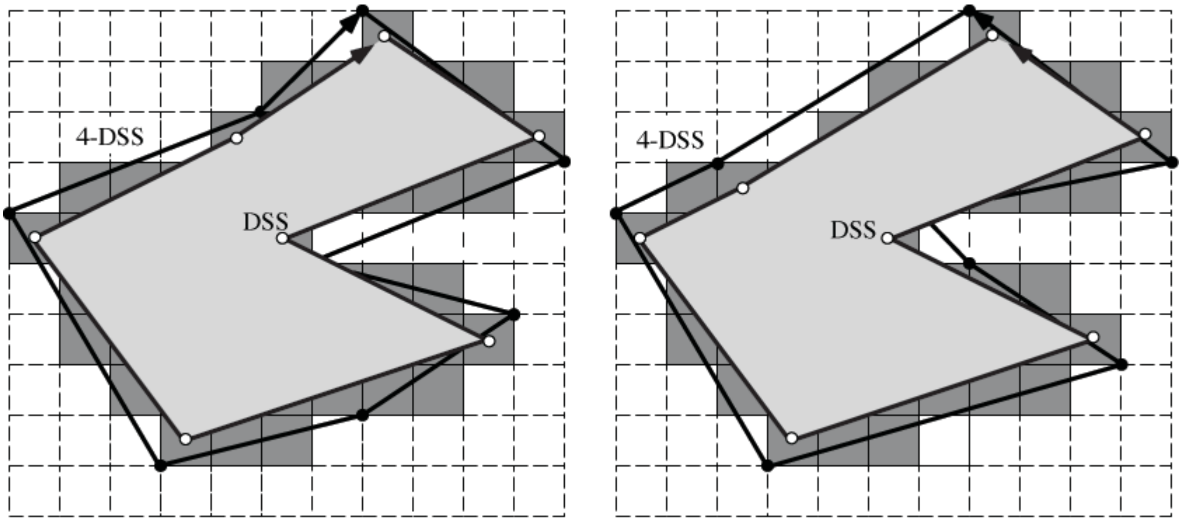
A clockwise and an anticlockwise traversal of the 4-border of a digital region that produce different segmentations into maximum-length DSSs. The start point of the traversal will also influence the result in general.

4-DSS versus DSS Segmentation

4-DSS: segmentation of a border cycle in the frontier grid

DSS: segmentation of a border cycle in the picture grid

In both cases we analyze the same region in the picture, but we obtain "slightly" (really?) different results.



Hypothesis 1: Area is larger and perimeter is larger in case of 4-DSS segmentation (frontier grid) compared to DSS-segmentation (picture grid).

Hypothesis 2: The number of resulting segments (for one region) is basically the same for both methods.

Question: Which method is closer to the true values when assuming (e.g.) Gauss digitization for measurable sets in the Euclidean plane?

Coursework

Related material in textbook: Sections 11.1.3 (just the material on 2D is sufficient) and 9.2 (the part on arithmetic geometry).

A.17. [7 marks][a] Page 11 states two hypothesis' and one question. Basically these are about differences in property measurements if applying either the cellular grid (frontier grid) or the picture grid (grid point model.

(i) Implement **K1990** and **DR1995** (note: there are free downloads on the internet).

(ii) Do experimental multigrid convergence analysis for area and perimeter estimates (assuming Gauss digitization) for at least one convex and one non-convex set (hint: originally given in the $[0, 1] \times [0, 1]$ square). (Hint: "Multigrid convergence analysis" means that a set is digitized with varying resolution, and we study the resulting error in property estimation with respect to increases in resolution.)

(iii) Discuss the two hypothesis' and the question stated on page 11.

[a] You can do either **A.16** or **A.17**, but not both.