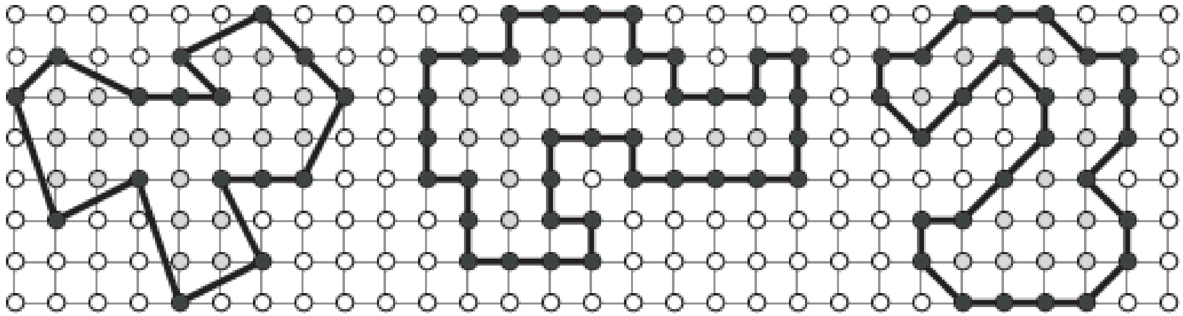
Pick's Formula

given: simple grid polygon containing α_0 grid points; l of those on its frontier



Left: $\alpha_0 = 36$ and $l = 16$. Middle: $\alpha_0 = 50$ and $l = 36$. Right: $\alpha_0 = 39$ and $l = 26$.

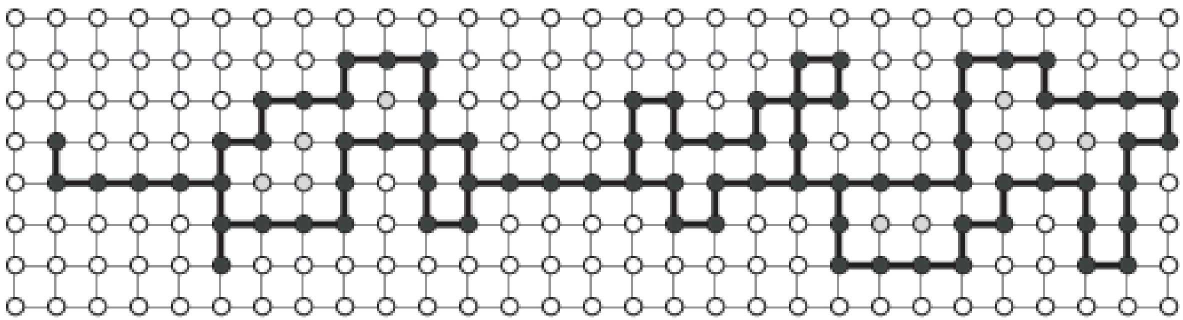
(G. Pick, 1899)

Theorem 1 *The area of a simple grid polygon equals* $f = \alpha_0 - \frac{l}{2} - 1$.

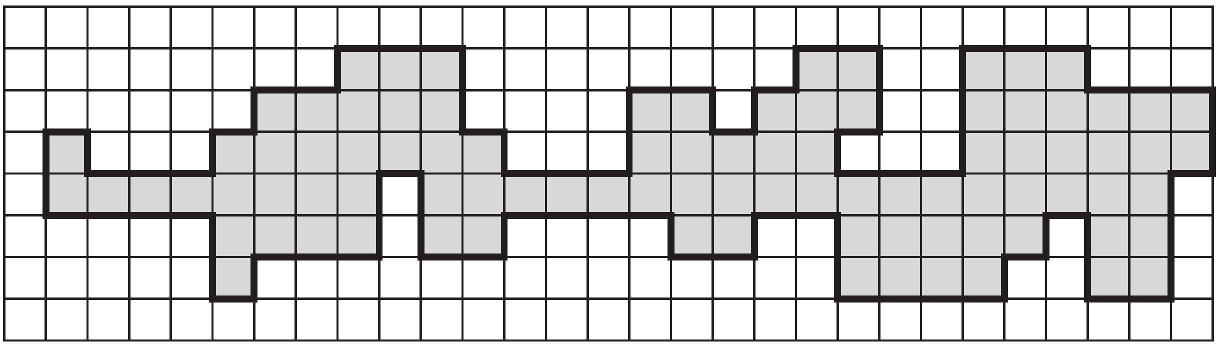
Examples above: $f = 27$ (left), $f = 31$ (middle), and $f = 25$ (right)

Decision for Grid Cell Model

Assume our task is to calculate the area of a region M, assuming the (unknown) preimage of this region was mapped by Gauss digitization into the grid, resulting into the picture as given:



The 4-border cycle circumscribes five simple isothetic grid polygons, and four arcs of width zero. A 2×2 square of grid points has area 1, and each arc has area 0. – It is more appropriate to consider pixels in the grid cell model, and to measure the area here (i.e., counting the pixels in the region):



We obtain a simple polygon (the *frontier* of M), and Pick's formula can be used. Vertices of 2-cells are now at grid point positions. The 29×8 picture grid (above) expanded into a 30×9 *frontier grid* (below).

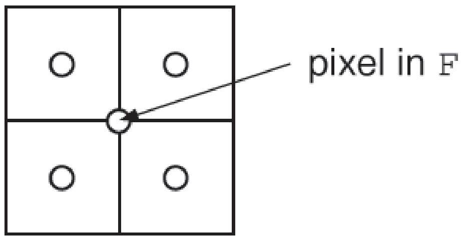
Frontier Grid

Assume a picture P defined on an $m \times n$ (picture) grid $\mathbb{G}_{m,n}$.

We analyze P in a picture F defined on an $(m+1) \times (n+1)$ frontier grid.

F is at the beginning like an empty "drawing board": we can fill it based on analyzing P.

A pixel in F corresponds to a grid vertex in P (assuming P in the grid cell model). Each grid vertex in P is incident with 4 grid squares (4 pixels).

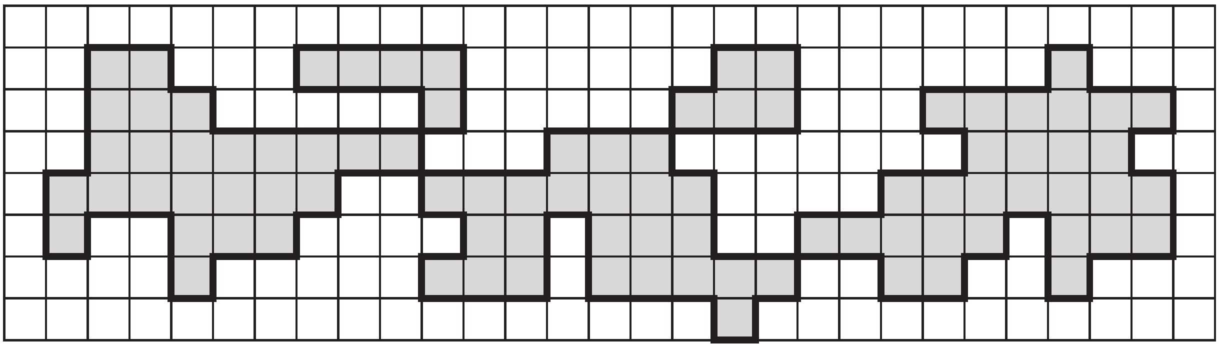


A value at a pixel in F will be defined based on a given analysis task for picture P.

Example: We trace in F the frontier of a 4-region M in P (see previous page). At each grid point $p = (x, y)$ in F, we have four pixels (and their values) in P which specify a neighborhood of p, and these are pixels $(x-1, y-1)$, $(x, y-1)$, $(x-1, y)$, and (x, y) in P. Depending on these values, we select the next isothetic step of a 4-path in F. Pixel values along this 4-path are changed into "black", and all others remain "white".

Non-Simple Polygons

For example, if applying s-adjacency (in P) then this may lead to a frontier which circumscribes several simple polygons:



Here we have five simple isothetic grid polygons.

General case: assume we circumscribe n simple grid polygons $\Pi_1, \ldots, \Pi_n$ ($n \geq 1$). The area is additive:

$$f = \mathcal{A}(\bigcup_{k=1}^{n} \Pi_k)$$

$$= (\alpha_0^{(1)} - \frac{l_1 - 1}{2} - 1) + \sum_{k=2}^{n-1} \left(\alpha_0^{(k)} - \frac{l_k - 2}{2} - 1 \right)$$

$$+ (\alpha_0^{(n)} - \frac{l_n - 1}{2} - 1)$$

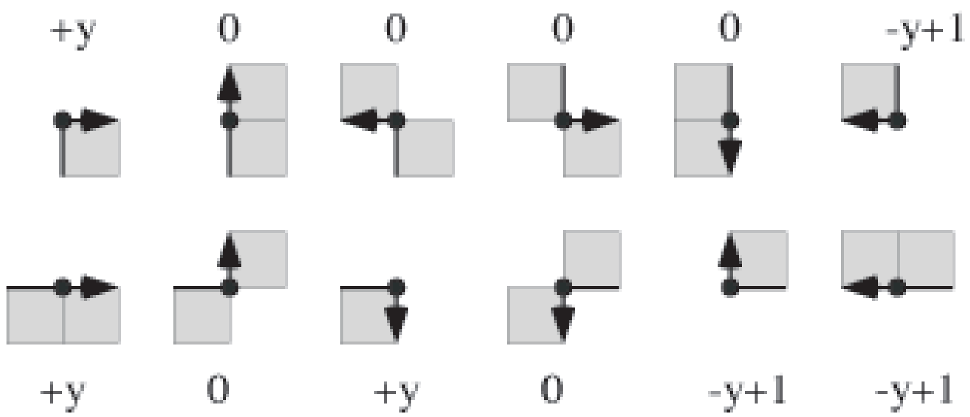
$$= \alpha_0 - \frac{L}{2} - 1$$

with $L = \sum_{k=1}^{n} l_k - 2(n-1)$ and $\alpha_0 = \sum_{k=1}^{n} \alpha_0^{(k)} - (n-1)$

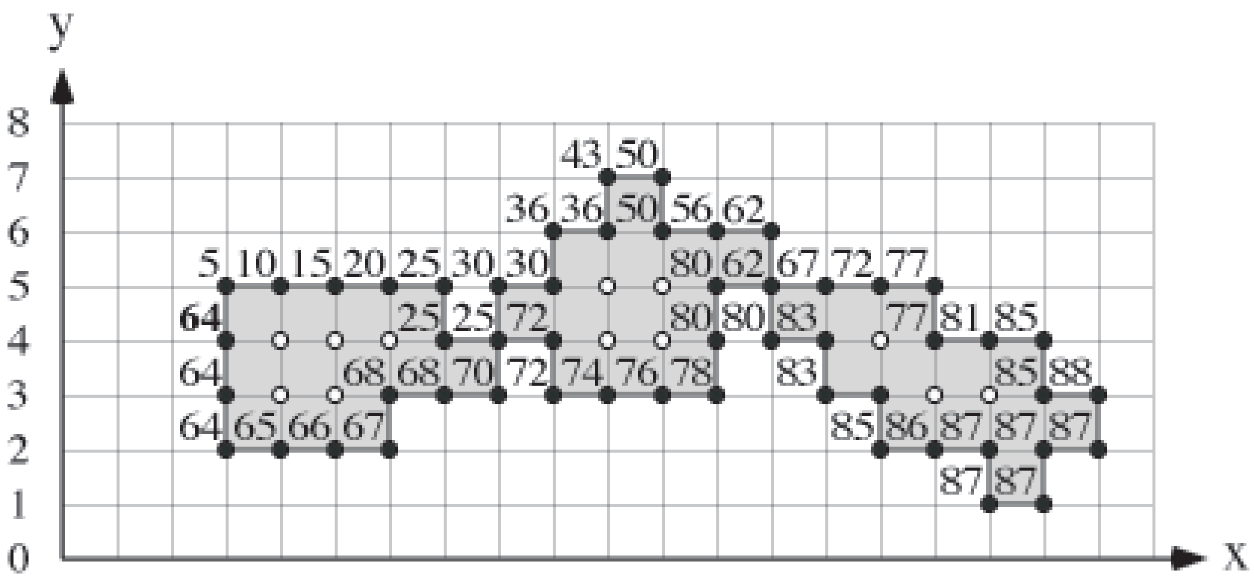
Note: L can be calculated when tracing the frontier, $n-1$ is the number of pixels in F which are visited twice. For α_0 see next page.

Discrete Column-Wise Integration

During a trace of a frontier in F, we arrive at a grid point (x, y) (in F) where the neighboring 4 pixels in P are either in the region (shown as a shaded square) or not.

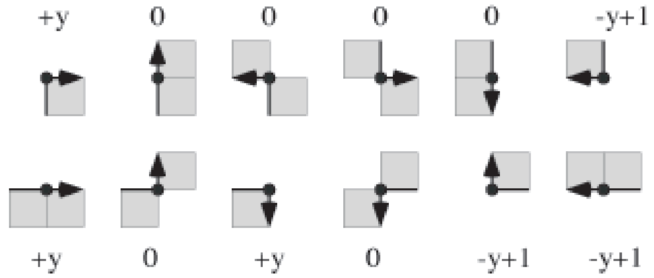


These are all the possible local patterns. Each pattern is labeled by its contribution (i.e., increment) for α_0.

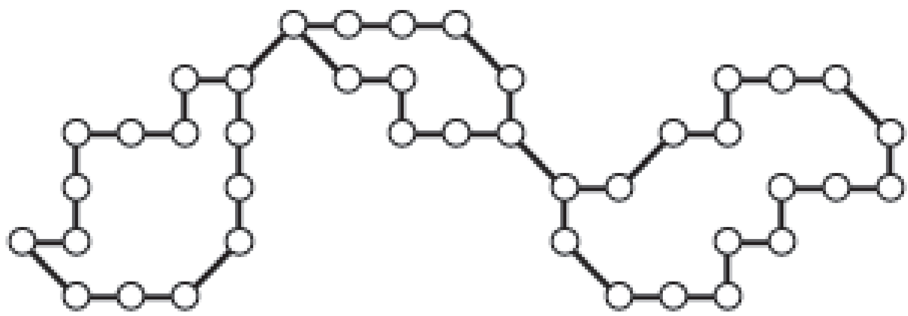


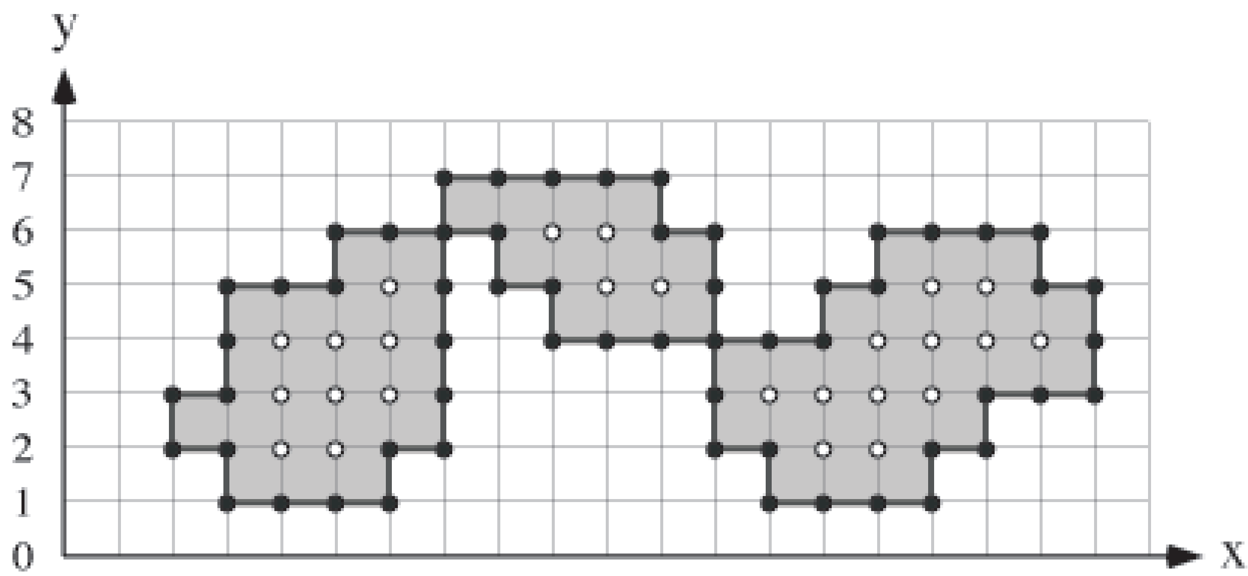
An example of counting α_0, starting at $(3, 5)$ with clockwise orientation. At the end we know that $\alpha_0 = 64$.

Another Example



Border tracing of a connected region M in P in s-adjacency:





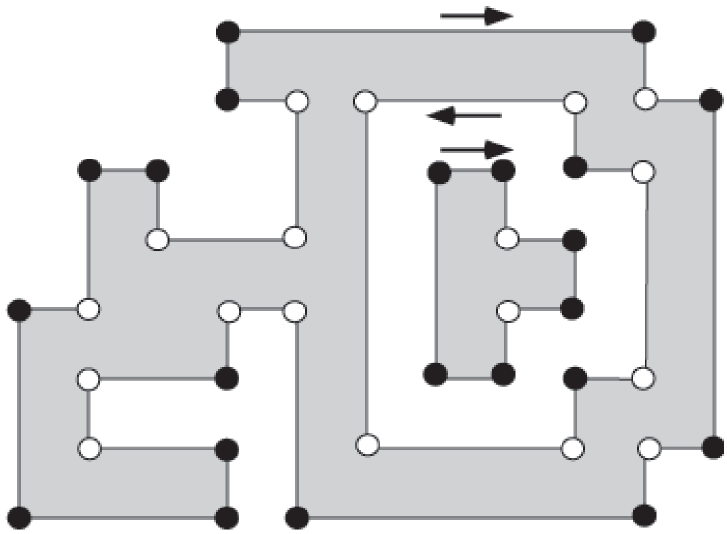
Frontier of M in the frontier grid:

$n =$ $L =$

$\alpha_0 =$ area =

Inner or Outer Border Cycle?

here we trace two outer and one inner border cycles (using, e.g., the algorithm from Lecture 10 if tracing borders of regions in $[\mathbb{Z}^2, A_4]$):



The resulting orientation (e.g., counter-clockwise for inner border cycles) is due to the algorithm.

The black vertices label *convex corners* (vertex angle is $\pi/2$), and the white vertices label *concave corners* (vertex angle is $3\pi/2$).

Theorem 2 *Let Π_A and Π_C be the numbers of convex and concave corners on a traced 4-border cycle ; then we have $\Pi_A - \Pi_C = 4$ for an outer border cycle, and $\Pi_A - \Pi_C = -4$ for an inner border cycle.*

Regular Oriented Adjacency Graphs

Regular oriented adjacency graph $\mathbb{G}_{\nu,\lambda} = [S, A, \xi]$: all $\nu(p) = \nu$ are constant, and all $\lambda(\rho) = \lambda$ are constant, where $p \in S$ and ρ any cycle generated by ξ

$$\nu = \frac{1}{\alpha_0} \sum_{p \in S} \nu(p) = 2\alpha_1/\alpha_0 \quad \text{and} \quad \lambda = \frac{1}{\alpha_2} \sum_{\rho} \lambda(\rho) = 2\alpha_1/\alpha_2$$

$$\text{thus} \quad \alpha_0/\alpha_1 = 2/\nu \quad \text{and} \quad \alpha_2/\alpha_1 = 2/\lambda, \text{ and}$$

$$2/\nu \ + \ 2/\lambda = 1 \ + \ 2/\alpha_1 \tag{1}$$

Examples: regular tilings of some surface (e.g., sphere or torus) define finite regular oriented adjacency graphs;

For infinite graphs, Equation (1) has only three integer-valued solutions: $\nu = \lambda = 4$; $\nu = 3$ and $\lambda = 6$; and $\nu = 6$ and $\lambda = 3$.

Let $S = \mathbb{Z}^2$ in these three infinite planar $\mathbb{G}_{\nu,\lambda}$s. All three infinite regular oriented adjacency graphs are planar; we have

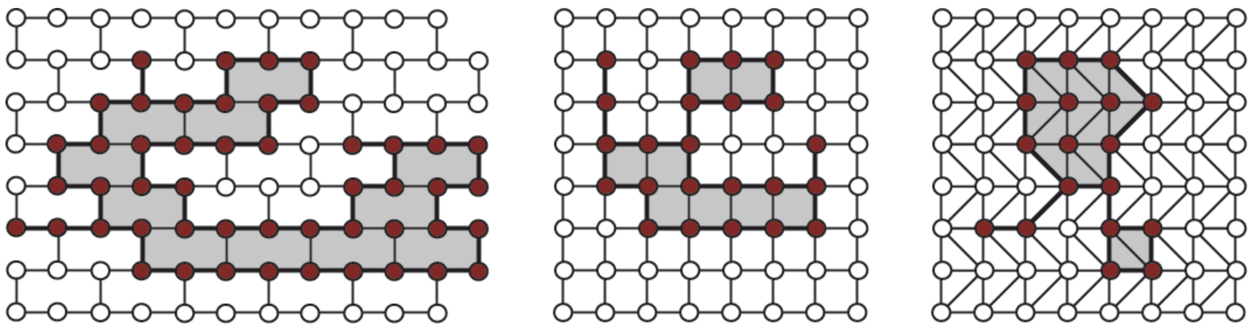
$$\alpha_0 - \alpha_1 + \alpha_2 = 2$$

Simply Connected Regions

Consider a finite subgraph of one of these three $\mathbb{G}_{\nu,\lambda}$s defined by a finite subset $M \subseteq \mathbb{Z}^2$.

Definition 1 *M is* simply connected *iff it has only one border cycle.*

Assume the border cycle has length l (= number of nodes on it). Let k be the number of invalid undirected edges between $\overline{M}$ and M, and f the number of atomic cycles of M (the "area" of M).



$\nu = 3, \lambda = 6, \alpha_0 = 49, \alpha_1 = 59, \alpha_2 = 12, l = 52, k = 29, f = 11$ (left);

$\nu = 4, \lambda = 4, \alpha_0 = 23, \alpha_1 = 30, \alpha_2 = 9, l = 28, k = 32, f = 8$ (middle);

$\nu = 6, \lambda = 3, \alpha_0 = 18, \alpha_1 = 32, \alpha_2 = 16, l = 19, k = 44, f = 15$ (right)

Simply Connected Regions

Altogether we have:

$$\begin{aligned} \alpha_0 - \alpha_1 + \alpha_2 &= 2 \\ \nu\alpha_0 - k &= 2\alpha_1 \\ \lambda(\alpha_2 - 1) + l &= 2\alpha_1 \end{aligned}$$

It follows that:

$$\nu l - \lambda k + \nu\lambda = (2\nu + 2\lambda - \nu\lambda)\alpha_1 = 0 \cdot \alpha_1 = 0$$

ν and λ are constants (of $\mathbb{G}_{\nu,\lambda}$).

A Generalization of Pick's Formula

(K. Voss, 1986: a generalization of Theorem 1)

Theorem 3 *For a region M of an infinite planar $G_{\nu,\lambda}$ that has no proper holes, we have $\alpha_0 = \lambda f/\nu + l/2 + 1$, where l is the length of the outer border cycle of M.*

General Case

(K. Voss, 1986: a generalization of Theorem 2)

Theorem 4 *For a region M of an infinite planar $G_{\nu,\lambda}$ and any of its border cycles, we have $k = \pm\nu + \nu l/\lambda$ where the outer border cycle has the positive sign and any inner border cycle has the negative sign.*

Suppose that M has $r \geq 1$ border cycles. Let L be the total length of all border cycles, and K the total number of all invalid edges assigned to these border cycles:

$$r = 2 + L/\lambda - K/\nu$$

r is a topologic invariant of M: L and K can be accumulated by examining all 4-neighborhoods of points in M; border cycle tracing is not necessary

Coursework

Related material in textbook: Sections 8.1.6 and 4.3.7. Solve Exercise 13 on page 305.

A.12. [6 marks] Design and implement (two variants of) an algorithm for discrete column-wise integration in the picture grid, such that the area (= number of grid points contained in a simply connected region) is calculated in $\mathcal{O}(l)$ time when tracing the l pixels on the (outer) border cycle of

(i) a 4-region (only isothetic steps on the border cycle), or

(ii) an 8-region (also allowing diagonal steps on the border cycle).

Apply both variants of the algorithm to digitized disks (Gauss digitization, grid resolution, e.g., between $h = 32$ and $h = 1024$) and

(iii) discuss the behavior of relative errors (compared to the true area of the disk) for increases in h; in particular,

(iv) with respect to Theorem 1 in Lecture 06?