

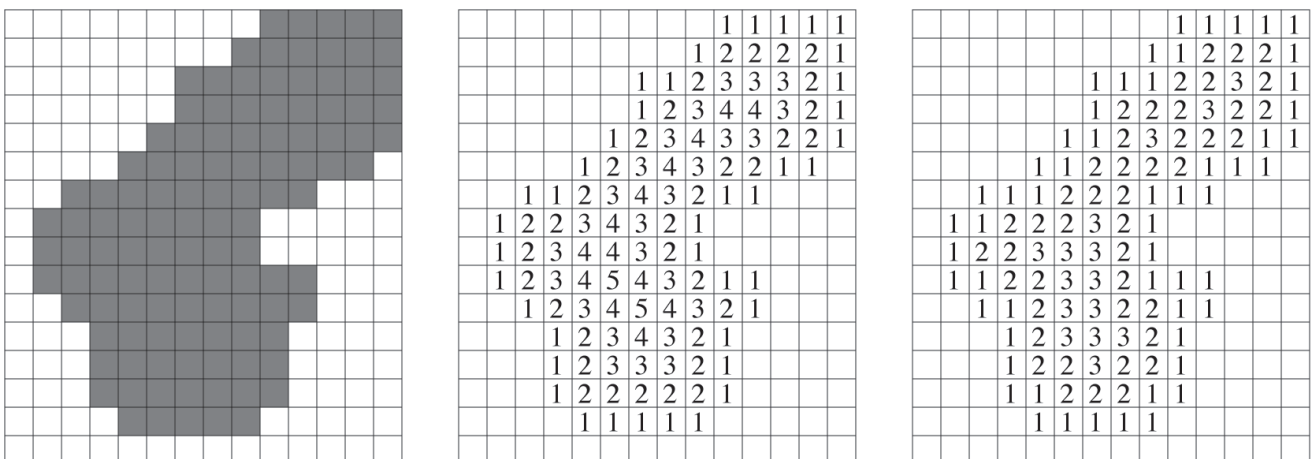
Distance Transforms

Let P be a binary picture (defined on grid \mathbb{G}) in which

$$\langle P \rangle = \{p : p \in \mathbb{G} \wedge P(p) = 1\} = P^{-1}(1), \text{ and}$$

$$\langle \bar{P} \rangle = \{p : p \in \mathbb{G} \wedge P(p) = 0\} = P^{-1}(0)$$

are proper subsets of \mathbb{G} . For any grid metric d_α , the d_α distance transform of P associates with every pixel p of $\langle P \rangle$ the d_α distance from p to $\langle \bar{P} \rangle$.



Left: Picture. Center: d_4 distance transform. Right: d_8 distance transform. – Distance transforms are frequently used when analyzing regions or patterns in pictures.

We assume that pixels of the background component (i.e., containing all pixels outside of the rectangular region \mathbb{G}) all have value 0.

Two-Pass Algorithm

The d_4 or d_8 distance transform of P (and others – see later) can be computed by performing a series of local operations while scanning \mathbb{G} twice.

(A *local operation* gives each pixel p a new value that depends only on the old values of the neighbors of p .)

For any $p \in \mathbb{G}$ let $B(p)$ (“before”) be the set of pixels (4- or 8-) adjacent to p that precede p when \mathbb{G} is scanned in standard order:

if p has coordinates (x, y) , B contains $(x, y + 1)$ and $(x - 1, y)$, and if we use 8-adjacency it also contains $(x - 1, y + 1)$ and $(x + 1, y + 1)$.

Let $A(p)$ (“after”) be the remaining (4- or 8-) neighbors of p .

First Scan

$$f_1(p) = \begin{cases} 0 & \text{if } p \in \langle \bar{P} \rangle \\ \min \{f_1(q) + 1 : q \in B(p)\} & \text{if } p \in \langle P \rangle \end{cases}$$

Compute $f_1(p)$ for all $p \in \mathbb{G}$ in a single standard scan of \mathbb{G} ; for each p , f_1 has already been computed for all of the qs in $B(p)$

(If p is on the top row or in the left column of \mathbb{G} , some of these qs are outside \mathbb{G} with $f_1 = 0$.)

2D Approximations to Euclidean Metric

Set of points within a given d_4 or d_8 distance from a given point is a square (and not a digitized disk).

These distances depend on direction; their “disks” are not good approximations to Euclidean disks.

If we restrict d_4 and d_8 to \mathbb{Z}^2 the set of grid points q such that $d_4(p, q) \leq k$ is a diagonally oriented square (a *diamond*) of odd diagonal length $2k + 1$ centered at p , and the set of grid points q such that $d_8(p, q) \leq k$ is an upright square of odd side length $2k + 1$ centered at p .

Example of a better approximation of the Euclidean metric:

$$d(p, q) = \max\{d_8(p, q), \frac{2}{3} \cdot d_4(p, q)\}$$

The set of grid points such that $d(p, q) \leq k$ is the intersection of an upright square of side length k with a diamond of diagonal length $3k/2$; this intersection is an upright octagon.

Best approximation of the Euclidean metric:

$\lceil d_e \rceil$ is the integer-valued metric that best approximates d_e .

“Incremental” algorithms for distance computation on a grid normally use local neighborhoods; this makes it easy to compute metrics such as d_4 , or d_8 or octagonal metrics, but not (in the same straightforward way) $\lceil d_e \rceil$.

2D Chamfer Metrics

A general method of defining approximations to Euclidean distance: count moves in different directions (e.g., isothetic moves, diagonal moves) and use different weights for these moves (e.g., 1 and $\sqrt{2}$).

Let $p, q \in \mathbb{Z}^2$ and ρ a sequence of king's moves from p to q .

m = number of isothetic moves; n = number of diagonal moves

$$l_{a,b}(\rho) = ma + nb$$

$$d_{a,b}(p, q) = \min_{\rho} l_{a,b}(\rho)$$

$d_{a,b}$ is the (a, b) chamfer distance (or weighted distance) from p to q (G. Borgefors, 1984).

Theorem 2 If $0 < a \leq b \leq 2a$ (called the Montanari condition), then the (a, b) chamfer distance $d_{a,b}$ is a metric.

The chamfer distance $d_{1,b}$ that best approximates d_e has $b = (1/\sqrt{2}) + \sqrt{\sqrt{2} - 1} \approx 1.351$; for this $d_{1,b}$ we have a maximum error

$$|d_e - d_{1,b}| \leq ((1/\sqrt{2}) - \sqrt{\sqrt{2} - 1})k \approx 0.06k$$

on an $(k + 1) \times (k + 1)$ grid \mathbb{G} . This optimal b is close to $4/3$; we therefore get a good approximation to $3d_e$ by using $a = 3$ and $b = 4$.

3D Chamfer Metrics

$d_{a,b,c}$ where a , b , and c correspond to moves in which

only one coordinate changes (isothetic moves),

two coordinates change,

and all three coordinates change,

and we can obtain good approximations to Euclidean distance by appropriately choosing a , b , and c .

Generalization

Generalized chamfer distances can be defined using additional types of moves that are not necessarily moves between 8-neighbors or 26-neighbors.

Conclusion

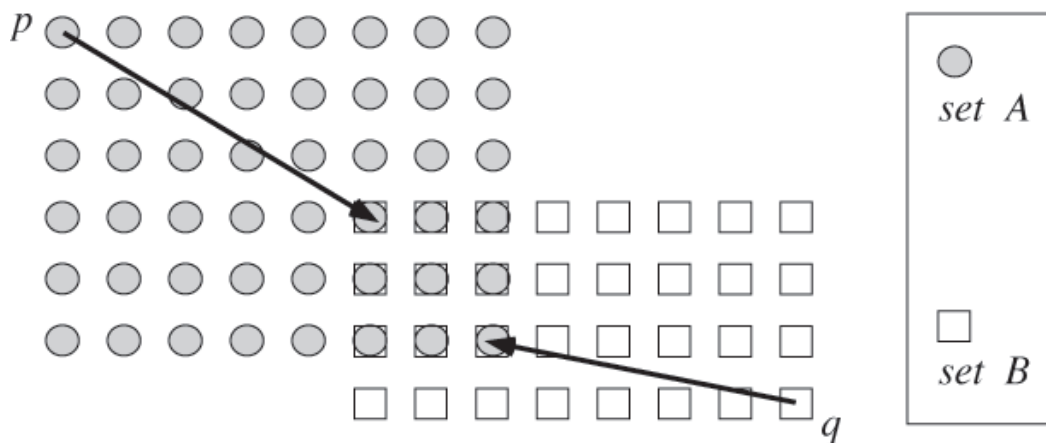
The two-scan algorithm can be used for any chamfer-metric distance transform, and this allows to “approximate the Euclidean case”.

Hausdorff Distance Between Sets

Any metric d on a set S can be extended to a *Hausdorff metric* on the family of all nonempty subsets A, B of S :

$$d(A, B) = \max \left\{ \max_{p \in A} \min_{q \in B} d(p, q), \max_{p \in B} \min_{q \in A} d(p, q) \right\}$$

(Actually, in general we should use sup and inf in this definition; but in our context we can stay with max and min.)



The Hausdorff distance between A and B is defined by the maximum of the two indicated distances.

Hausdorff Metric in Steps

In general: *closest distance* from $p \in S$ to $T \subseteq S$

$$d(p, T) = \inf_{q \in T} d(p, q)$$

Let $A, B \subset S$, $h_p(B) = d(p, B)$ for all $p \in A$, $h_p(A) = d(p, A)$ for all $p \in B$, and

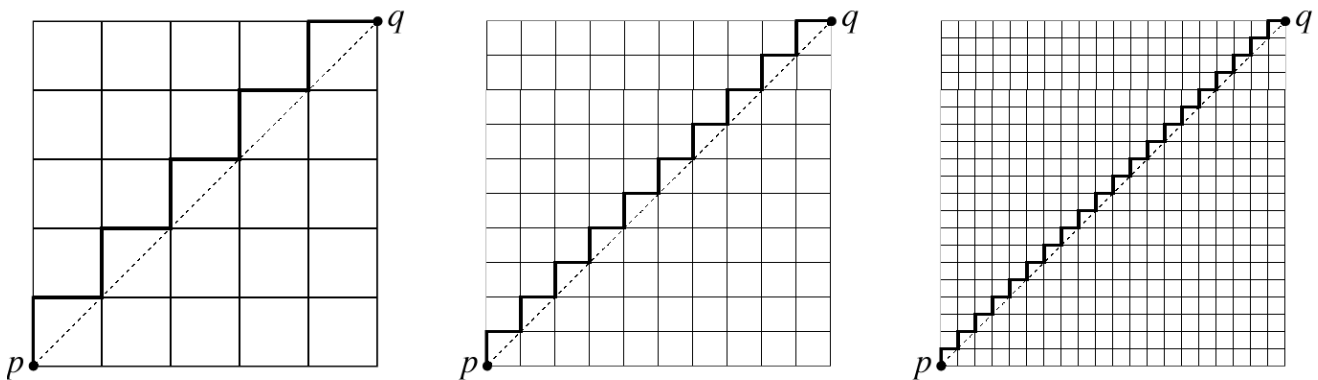
$$h_A(B) = \sup_{p \in A} h_p(B); \quad h_B(A) = \sup_{p \in B} h_p(A)$$

Then we have the Hausdorff metric

$$d(A, B) = \max\{h_A(B), h_B(A)\}$$

Figure on page 7: Assume $d = d_e$. From $h_A(B) = h_p(B) = \sqrt{34}$ and $h_B(A) = h_q(A) = \sqrt{26}$ we have $d(A, B) = h_A(B)$.

Convergence in Hausdorff Metric



The Hausdorff distance between diagonal pq (set A) and “staircase arc” (set B) converges to zero as the grid resolution increases (but the length of the staircase does not converges to the length of the diagonal).

Definition Based on Dilation

An alternative way of defining the Hausdorff metric uses the ε -neighborhood of a set:

$$U_\varepsilon(A) = \{q : q \in S \wedge h_q(A) < \varepsilon\}$$

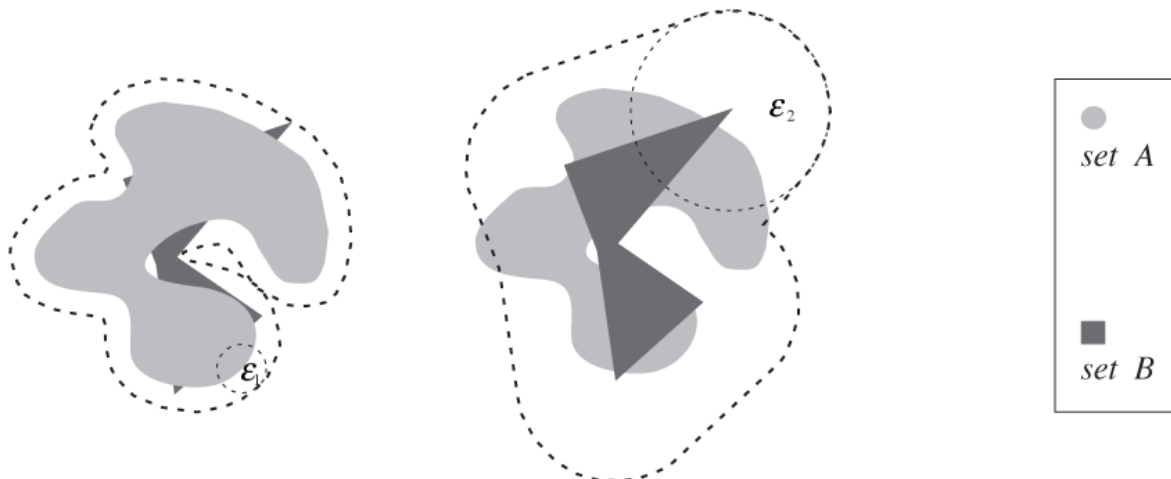
where h_q is defined by a metric d on S , $\varepsilon > 0$, and $A \subset S$.

Let A and B be nonempty subsets of S :

$$h_A(B) = \inf\{\varepsilon : A \subseteq U_\varepsilon(B)\}$$

(analogously for $h_B(A)$) and

$$d(A, B) = \max\{h_A(B), h_B(A)\}$$



Left: B (a simple polygon) is completely contained in $U_{\varepsilon_1}(A)$.
 Right: A is not completely contained in $U_{\varepsilon_2}(B)$, showing that $d(A, B) > \varepsilon_2$.

An Algorithm for Hausdorff Distance

Assume two finite sets $A, B \subset \mathbb{G}_{m,n}$ of grid points.

For any $S \subset \mathbb{G}_{m,n}$, the *distance field* $F(S)$ is an array of size $m \times n$ such that $F(S)(p) = h_p(S)$; in particular, $F(S)(p) = 0$ iff $p \in S$.

Note: the distance field can be calculated (in $\mathcal{O}(mn)$ computation steps) by a two-scan distance transform, assuming that d is a chamfer metric.

1. Calculate a distance field $F(A)$ in an array of size $m \times n$.
2. Calculate a distance field $F(B)$ in an array of size $m \times n$.
3. Let a be the maximum value in $F(A)$ at all positions belonging to B .
4. Let b be the maximum value in $F(B)$ at all positions belonging to A .
5. $H(A, B) = \max\{a, b\}$.

Algorithm for calculating the Hausdorff distance between two subsets A and B of an $m \times n$ grid.

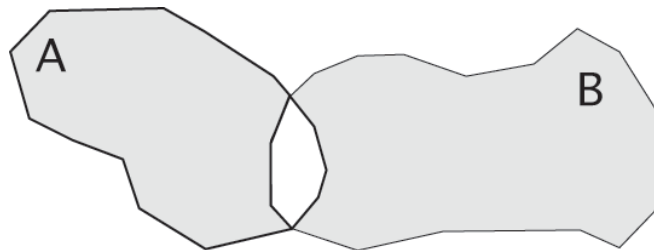
Metrics by Symmetric Difference

Hausdorff metrics are based on maximum distances between sets; a single point (an “outlier”) in a set can strongly influence these distances.

Distances between sets defined by set-theoretic differences are less sensitive to single points.

Symmetric difference between two subsets A, B of a set S :

$$A\Delta B = (A \setminus B) \cup (B \setminus A)$$



The symmetric difference is shaded.

$$d_{\text{sym}}(A, B) = \text{card}(A\Delta B) \quad \text{and} \quad d'_{\text{sym}}(A, B) = \frac{\text{card}(A\Delta B)}{\text{card}(A \cup B) + 1}$$

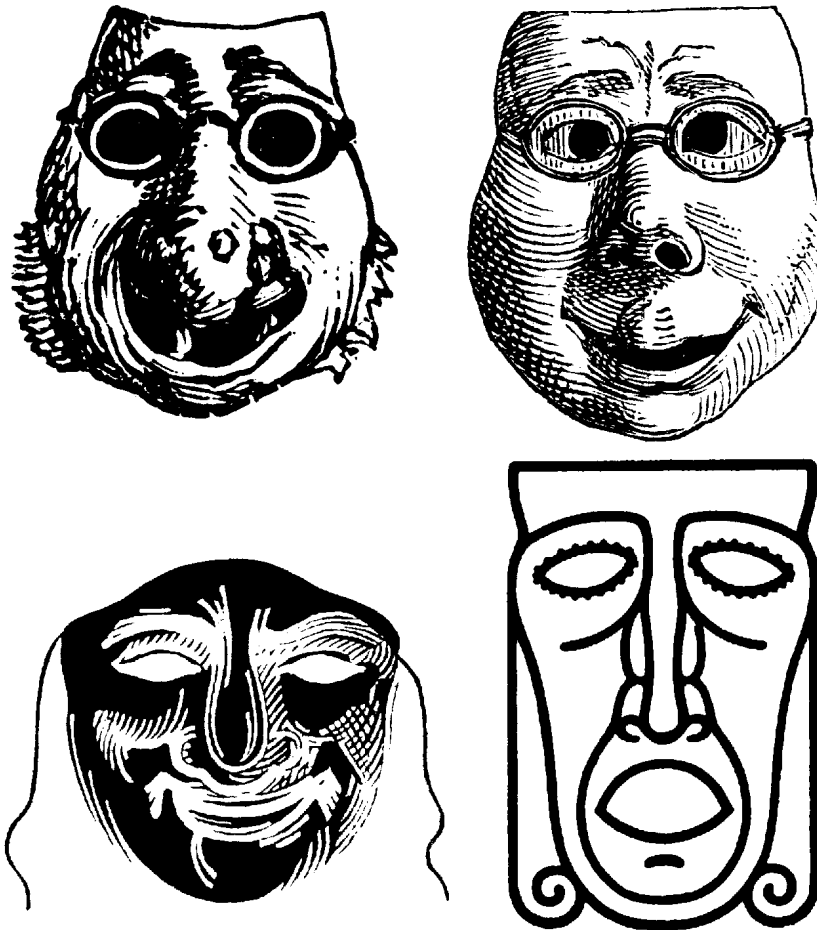
Theorem 3 d_{sym} and d'_{sym} are metrics on the family of all finite subsets of S .

Here, we only have to calculate cardinalities $\text{card}(M)$ of sets M of pixels or voxels.

Coursework

Related material in textbook: Sections 3.4.2, 3.1.8, 3.2.3 (you may skip d_h), and 3.2.5. Do Exercises 11 and 17 on page 114.

A.8. [4 marks] Use a set of at least four different binary pictures of similar objects. Implement a program which measures the similarity of these based on calculated distances.



For example, at first center all objects at their origin (possibly align them also in orientation) and then compare these objects based on a chosen metric.