Variable Grid Resolution

Pictures are given at one particular grid resolution. There might be alternative ways for generating pictures (showing, e.g., the same scenery) at different grid resolutions (e.g., by changing parameters of a flatbed scanner, or by generating a *picture pyramid*). Tests or assumptions of varying grid resolution are often a useful way of analyzing picture analysis procedures.

grid constant θ = the distance between neighboring grid lines (i.e., $\theta > 0$ is a real)

grid resolution h = the inverse of the grid constant; it refers to the number of grid elements per unit of distance without specifying the physical size of the unit (i.e., $h > 0$ is an integer).

example: grid resolution $h = 1$, then either one grid point in a unit or two grid points as endpoints of a unit

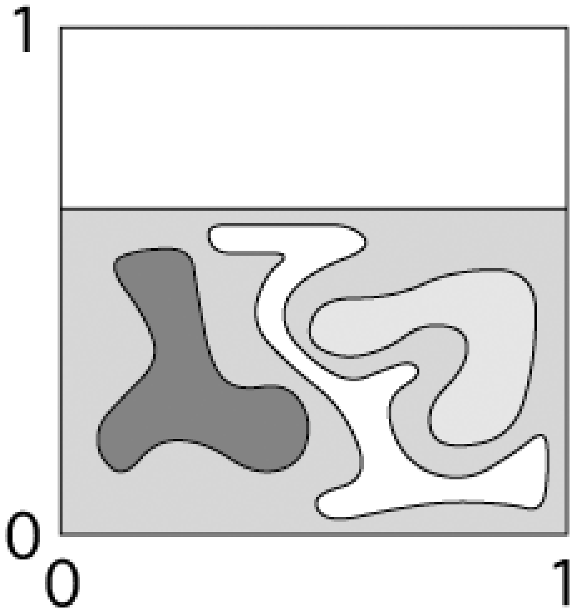
in general: maximum number of grid points per unit is $h + 1$

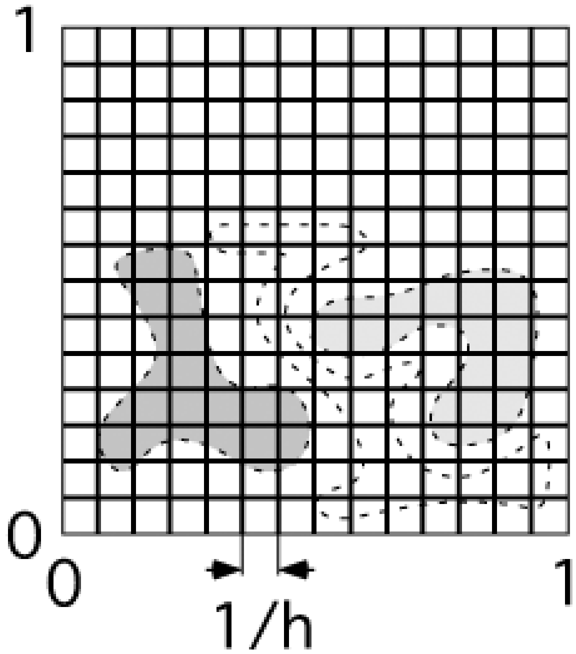
Let $\mathbb{Z}_h = \{i/h : \ i \in \mathbb{Z}\}$.

$\mathbb{Z}_h^2$ is the set of all 2D grid points in a grid of resolution $h > 0$.

$\mathbb{Z}_h^3$ is the set of all such 3D grid points.

Grid Resolution Model





an original picture is assumed to be in (a rectangular subset of) the unit square $[0, 1] \times [0, 1]$

a digital picture is defined by some process of digitization, which maps the original picture into a grid of resolution h

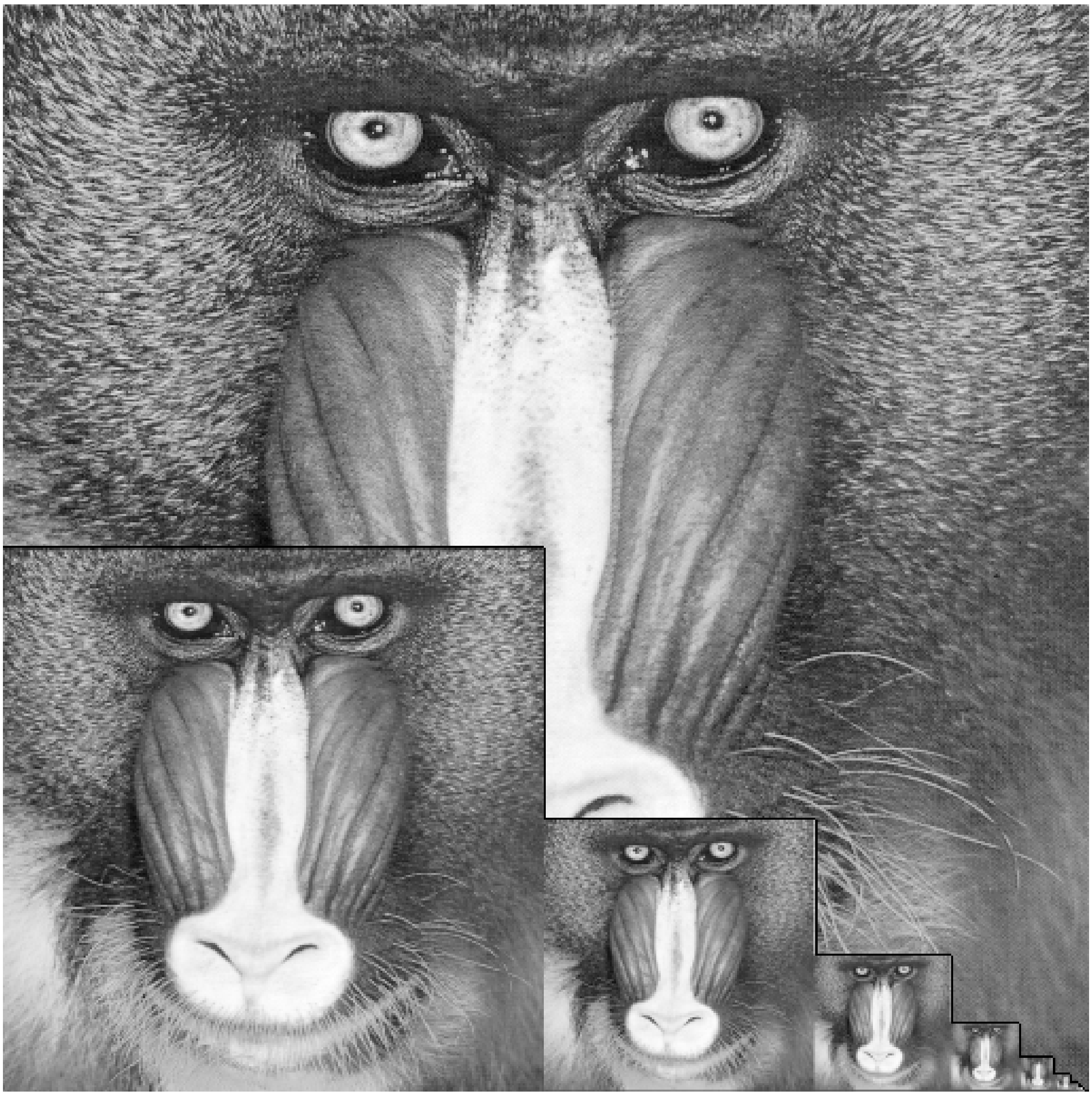
if implementing picture analysis procedures, it is common to ignore the value of h (i.e., we just assume $h = 1$ and a grid in $\mathbb{Z}^2$)

studies depending on resolutions take h into account

Example of A Picture's Pyramid

the basic layer of the pyramid is the original picture of size $2^n \times 2^n$ (i.e., the picture at grid resolution $h = 2^{-n}$)

a layer of size $2^m \times 2^m$ is mapped into the next layer of size $2^{m-1} \times 2^{m-1}$ by mapping disjoint windows of 2×2 pixels into a single pixel (e.g., by taking the mean of all four pixel values)

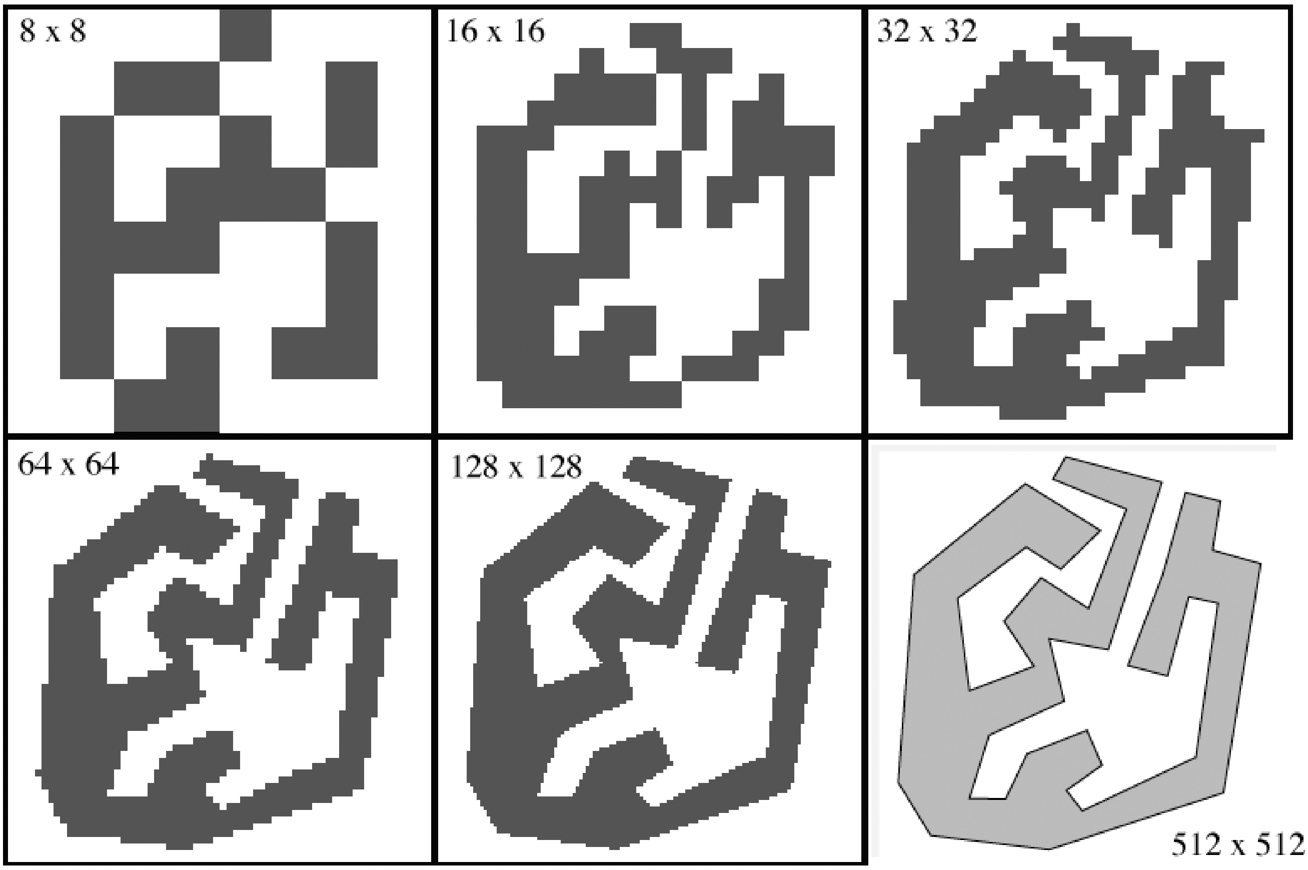


top of pyramid: a 1×1 picture of resolution $h = 1$

C. F. Gauss (1777–1855) studied the measurement of the area of a planar set $S \subset \mathbb{R}^2$ by counting the grid points $(i, j) \in \mathbb{Z}^2$ contained in S. This approach suggests the following:

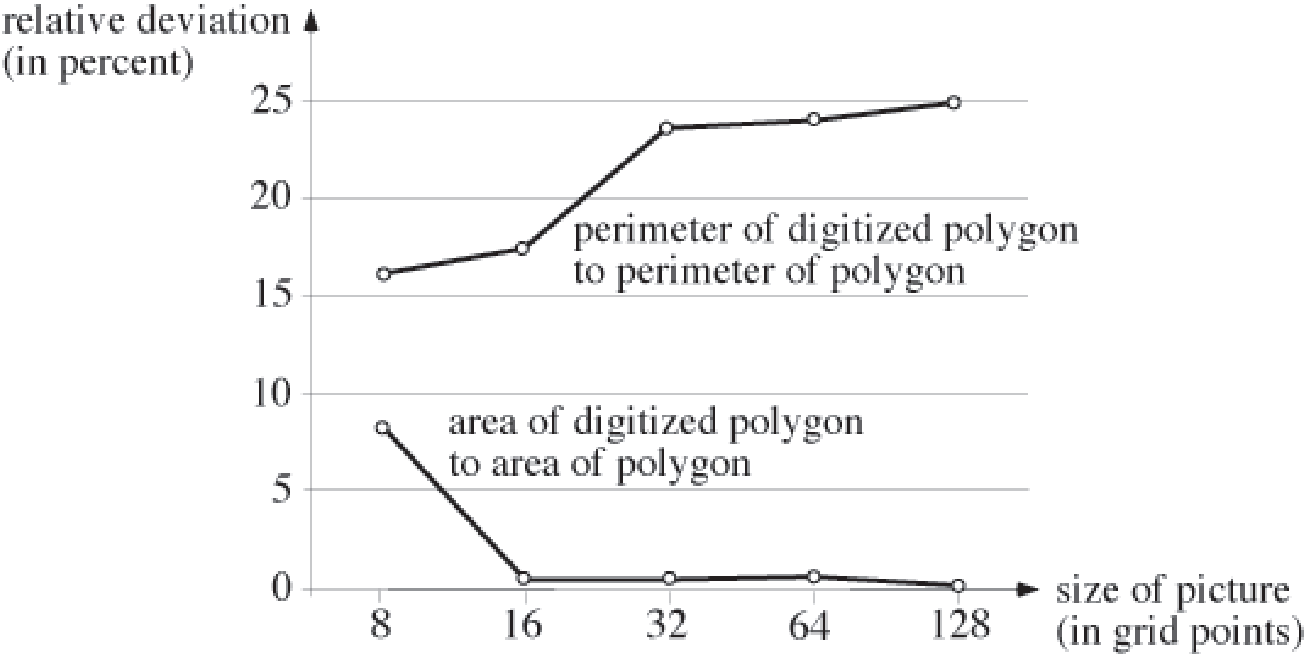
Gauss Digitization

Let S be a subset of the plane. The *Gauss digitization* $G(S)$ is the union of the grid squares with center points in S (in general: $G_h(S)$, if a grid of resolution h is used).



Gauss digitization $G_h(\Pi)$ of a simple polygon Π using grids of resolutions $h = 8$ to $h = 128$; original polygon (basic layer of pyramid) was drawn on a grid of resolution $h = 512$

The original polygon Π has area 102,742·5 and perimeter 4,040·7966…if drawn on a 512×512 grid.



Relative deviations of the area and perimeter of $G_h(\Pi)$ from those of Π when Π is digitized on a $2^n \times 2^n$ grid (i.e., $h = 2^n$).

> The *relative deviation* is the absolute difference between the property values for $G_h(\Pi)$ and Π divided by the property value for Π. The *perimeter* of $G_h(\Pi)$ is the number of 1-cells on its frontier times $1/h$, and the *area* of $G_h(\Pi)$ is the number of 2-cells in $G_h(\Pi)$ times $1/h^2$.

As discussed in Lecture 05: the perimeter of $G_h(\Pi)$ is not an acceptable estimate of the perimeter of Π (here: the relative error seems to converge to 25%).

However, the area of $G_h(\Pi)$ seems to be a "pretty accurate" estimator for the area of Π! Already at $h = 16$, the relative error is close to zero!

Content Estimation

Gauss digitization is defined analogously in 3D. If $S \subset \mathbb{R}^3$, the Gauss digitization $G_h(S)$ is the union of all of the 3-cells (in a grid of resolution $h > 0$) with center points in S.

A grid polyhedron is *simple* iff it is topologically equivalent to a closed sphere.

The surface area $\mathcal{S}(\Pi)$ and the volume $\mathcal{V}(\Pi)$ of a simple grid polyhedron Π are defined by the number of 2-cells that form the frontier of Π multiplied by h^{-2} and the number of 3-cells contained in Π multiplied by h^{-3}, respectively.

Let $S \subset \mathbb{R}^n$ $(n \geq 1)$ be a closed bounded set that has measurable *content* $\mathcal{C}(S)$, which is the length $\mathcal{L}(S)$ for $n = 1$, the area $\mathcal{A}(S)$ for $n = 2$, and the volume $\mathcal{V}(S)$ for $n = 3$.

How to use grid polygons or grid polyhedra (i.e., the available data in picture analysis) for estimating such properties of the unknown original set?

Studies in Number Theory

(just to tell some pointers to related mathematical studies)

$S \subset \mathbb{R}^n$, $h > 0$ (our interest: $n = 2$ or $n = 3$)

magnification of S *by factor* h for $n = 2$:

$$S_h = h \cdot S \ = \ \{(h \cdot x, h \cdot y) : \ (x, y) \in S\}$$

(i.e., this leaves the origin $(0, 0)$ fixed)

$\mathcal{N}(S) = \mathcal{C}(G(S))$ = number of grid points (note: for $h = 1$) in S, defined either for $n = 2$ or $n = 3$ by its Gauss digitization

suppose: S_h depends on only one parameter $h > 0$ (e.g., a disk or a sphere of radius h)

$h \to \infty$, then $\quad \mathcal{N}(S_h) = \mathcal{C}(S_1) \cdot h^n + \mathcal{O}(h^{n-1})$

> $\mathcal{O}(\ldots)$ is the “big-Oh” as known from discussions of time complexities of algorithms.

H. Steinhaus (1947) for 2D: $\;|\mathcal{N}(S) - \mathcal{C}(S)| \leq 4(\mathcal{P}(S) + 1)$ (where we assume that content $\mathcal{C}(S)$ and perimeter $\mathcal{P}(S)$ are defined)

> (For further results, see E. Landau [1955] and H. Davenport [1951] in the Bibliography of the textbook.)

C. F. Gauss and his colleague P. Dirichlet (1805–1859) at Göttingen University already knew that the number of grid points $(i, j) \in \mathbb{Z}^2$ inside $h \cdot S$, where S is a planar convex set, estimates the area $\mathcal{A}(h \cdot S)$ within an asymptotic order of $\mathcal{O}(\mathcal{P}(h \cdot S))$.

Note: S is convex, thus the perimeter $\mathcal{P}(h \cdot S)$ is $\mathcal{O}(h)$.

Using the Grid Resolution Model

this "translates" from the magnification model (of number theory, see E. Krätzel [1981]) into the grid-resolution model of picture analysis as follows:

Theorem 1 *For any planar convex set* S *and any grid resolution* $h > 0$, $|\mathcal{A}(G_h(S)) - \mathcal{A}(S)| = \mathcal{O}(h^{-1})$.

This result can be extended to nonconvex planar sets that can be partitioned into finite numbers of convex sets.

The theorem implies that counting grid points inside such an S provides a convergent estimate of $\mathcal{A}(S)$ as the grid resolution h goes to infinity.

Formalized General Evaluation Scheme

Let $\mathbb{F}$ be a family of sets S in $\mathbb{R}^n$ (the "objects of interest", e.g. a class of arcs or curves, or a class of volumes in 3D space).

Let $dig_h(S)$ denote a digitization of S on a grid of resolution h (e.g., grid-intersection or Gauss digitization)

Assume that a property $\mathcal{Q}$ (e.g., area, perimeter) is defined for all $S \in \mathbb{F}$.

Definition 1 *An estimator $E_{\mathcal{Q}}$ is* multigrid convergent *for $\mathbb{F}$ and for dig_h iff, for any $S \in \mathbb{F}$, there is a grid resolution $h_S > 0$ such that the estimated value $E_{\mathcal{Q}}(dig_h(S))$ is defined for any grid resolution $h \geq h_S$, and*

$$|E_{\mathcal{Q}}(dig_h(S)) \; - \; \mathcal{Q}(S)| \leq \kappa(h)$$

where κ is a function defined on the real numbers that takes on only positive real values and converges to zero as $h \to \infty$.

The function κ specifies the *speed of convergence*. (Note: we had linear convergence in n in case of Archimedes' method of estimating π.)

Examples: $\kappa(h) = h^{-2}$ specifies *quadratic convergence*, and $\kappa(h) = h^{-1-a}$, $a > 0$, specifies *superlinear convergence*

Example 1 (discussed before):

$\mathbb{F}$ = class of circles, $dig_h(S)$ = grid-intersection digitization, $\mathcal{Q}$ = perimeter, and $E_{\mathcal{Q}}$ = the length of the resulting 8-curve

Result: there is no κ in this case (i.e., this is not a multigrid convergent estimator)

Example 2 (discussed before):

$\mathbb{F}$ = class of planar convex regions, $dig_h(S)$ = Gauss digitization, $\mathcal{Q}$ = area, and $E_{\mathcal{Q}}$ = the number of grid points in the digital set

Result: there is a function κ which is in the order of h^{-1} (i.e., this is a multigrid convergent estimator with [at least] linear convergence)

Example 3 (to be discussed later):

$\mathbb{F}$ = class of planar rectifiable curves, $dig_h(S)$ = grid-intersection digitization, $\mathcal{Q}$ = length of curve, and $E_{\mathcal{Q}}$ = the length of polygonal chains defined by DSS-approximation of the given curve

Result: there is a function $\kappa(h) \approx 4{\cdot}5/h$ (i.e., this is a multigrid convergent estimator with [at least] linear convergence, where even an absolute upper bound is known)

Two ways to study convergence with respect to increased grid resolution:

(i) way in number theory: consider magnified sets $h \cdot S$, but always digitized on the same grid with unit grid constant

(ii): way in numeric mathematics or picture analysis: keep set S at constant size and digitize it on grids with grid constant $1/h$:

> ('forward option') digitize a given curve in the unit square for increasing resolutions, or
>
> ('backward option') use a given high-resolution picture as the basic layer of a picture pyramid, and derive lower-resolution copies for the other layers of the pyramid.

In both cases, $h \to \infty$ corresponds to an increase in grid resolution.

This lecture uses (in general) approach **(ii)**. This is motivated by the assumed scenario in which the set to be analyzed remains physically the same while improvements in hardware (e.g., scanners, computing power) allow refinements in grid resolution.

Coursework

Related material in textbook: Sections 2.1.2, 2.3.1, and 2.4. and do Exercise 13 on page 72.

A.6. [5 marks] Do Exercise 12 on page 72. Note that the midpoint of the digitized disk is not at a grid point position in general for the varying values of h. We obtain a sequence of relative errors (for increases in h) which is not always monotonous in its behavior (i.e., monotonous decreasing or increasing). A possible option is to use the *sliding mean* of this sequence for obtaining a sequence of monotonous behavior. Of course, the calculated *Archimedes constant* (i.e., value of h where we achieve the same accuracy as Archimedes) will then depend upon the width of the sliding mean.

> Let $e_0, e_1, e_3, \ldots$ be a sequence of reals. The *sliding mean* is specified by one parameter, the width $w = 2k + 1$ which is a positive odd integer, and produces the sequence $m_k, m_{k+1}, m_{k+2}, \ldots$, where m_j is the mean of w values of the original sequence:

$$m_j = \frac{1}{2k+1} \sum_{i=j-k}^{j+k} e_i$$

For example, you may start your experiments with $w = 31$.