Definition of Length

Let ϕ be a parameterized continuous path $\phi : [a, b] \to \mathbb{R}^2$ such that $a \neq b$, $\phi(a) = \phi(b)$, and let $\phi(s) \neq \phi(t)$ for all s, t ($a \leq s < t < b$).

C. Jordan defined in 1893 the following, today known as *Jordan curve* in the plane:

$$\gamma = \{(x, y) \ : \ \phi(t) = (x, y) \ \wedge \ a \leq t \leq b\}$$

A *Jordan arc* γ in the plane is defined by a subinterval $[c, d]$ where $a \leq c < d \leq b$.

A *rectifiable Jordan arc* γ has a bounded *arc length* as follows, where d_e is the Euclidean metric:

$$\mathcal{L}(\gamma) = \sup_{n \geq 1 \wedge c = t_0 < \cdots < t_n = d} \sum_{i=1}^{n} d_e\left(\phi\left(t_i\right), \phi\left(t_{i-1}\right)\right) \quad < \infty$$

Alternative Definition

In a first attempt, C. Jordan proposed in 1883 the following definition of a curve:

$$\gamma = \{(x, y) \ : \ x = \alpha(t) \ \wedge \ y = \beta(t) \ \wedge \ a \leq t \leq b\}$$

However, using such a parameterization, G. Peano defined, in 1890, a curve known as the *Peano curve* that fills the whole unit square. Despite that, this 1883 definition is in common use for arc length calculation. We assume differentiable functions α and β:

$$\mathcal{L}(\gamma) = \int_a^b \sqrt{\left(\frac{d\alpha(t)}{dt}\right)^2 + \left(\frac{d\beta(t)}{dt}\right)^2} \, dt$$

In picture analysis, we have to deal with curves that are given in digitized pictorial form and for which a parametric description is often not of interest. However, the true length of an arc or curve can be used to evaluate methodologies for measuring length in picture analysis.

A General Evaluation Scheme

1. Define a methodology for estimating the length of an arc or curve (possibly limited to a particular class of arcs or curves) assuming that this arc or curve is only given by a particular finite representation (e.g., a polygon).
2. Consider examples of arcs or curves in the class of interest where the true length is known; map these arcs or curves into finite representations (e.g., by grid-intersection digitization into an ordered sequence of grid points, or by sampling into a polygonal chain) as assumed in your methodology.
3. Apply your methodology for these finite representations and compare the estimated length against the true length.

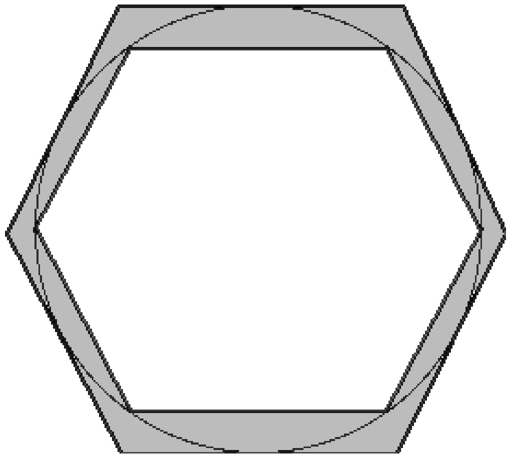
A consideration of various finite (i.e., of varying cardinalities) representations of the same curve allows to understand how the size of the discrete representation influences the accuracy of the estimation.

Estimation of π by Archimedes

(more than 2200 years ago)

class of curves: circles (i.e., curves of [today] known length $2\pi r$)

finite representation: inner and outer regular n-gon (An n-gon is called *regular* if its edges all have the same length.)



methodology: perimeter of inner n-gon as lower bound, and perimeter of outer n-gon as upper bound; in the figure we have $n = 6$, which gives

$$3 < \pi < 3{\cdot}46$$

example: $n = 96$ (this is the maximum number n as used by Archimedes), then

$$\frac{223}{71} < \pi < \frac{220}{70} \quad \text{(i.e.,} \quad \pi \approx 3{\cdot}14)$$

the logical start is at $n = 3$: the equilateral triangle

then, by dividing each side by its perpendicular bisector, we can construct (inner or outer) n-gons for $n = 6, 12, 24, 48, 96, \ldots$ (i.e., these are the varying representations of the same curve)

Inner n_m-gon P_{n_m}

perimeter $\mathcal{P}(P_{n_m}) = n_m \cdot e_m$, assuming $n_m = 3 \times 2^m$ edges of length e_m

$m = 0$ (i.e., the inner equilateral triangle): $e_0 = r\sqrt{3}$

$m = 1$ (i.e., the inner hexagon): $e_1 = r$

in general (for inner n_m-gons): $e_{m+1} = \sqrt{2r^2 - r\sqrt{4r^2 - e_m^2}}$

Outer n_m-gon Q_{n_m}

again, consider perimeter $\mathcal{P}(Q_{n_m}) = n_m \cdot f_m$, assuming $n_m = 3 \times 2^m$ edges of length f_m

$m = 0$ (i.e., the outer equilateral triangle): $f_0 = \ldots$

$m = 1$ (i.e., the outer hexagon): $f_1 = \ldots$

in general (for outer n_m-gons): $f_{m+1} = \ldots$

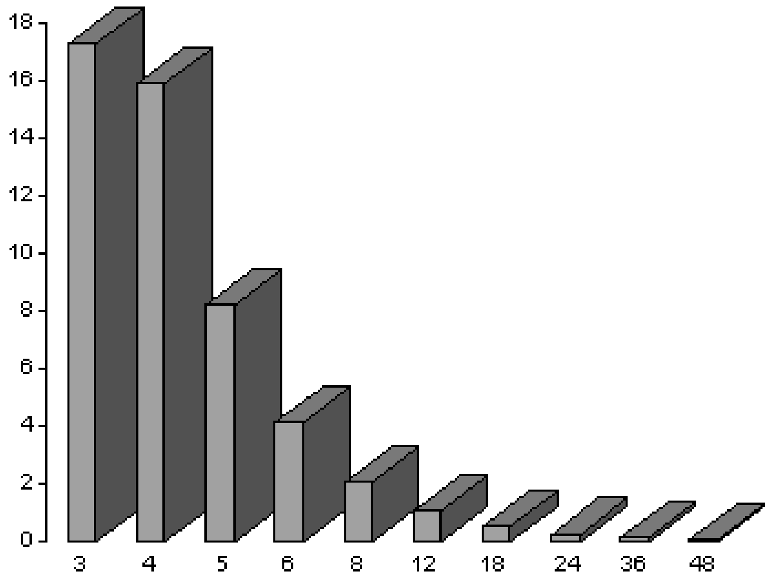
Let $\mathcal{P}(P_{n_m})$ be the perimeter of the inner n_m-gon. It follows that the *estimation error*

$$\kappa(n_m) = |\mathcal{P}(P_{n_m}) - 2\pi r| \approx \frac{2\pi r}{n_m}$$

converges to zero as $n_m \to \infty$

(see Appendix for further discussion and the original method of Archimedes)

To be more precise, the formula says that the *speed of convergence* $1/\kappa(n)$ is (asymptotically) a linear function of n.



The figure shows percentage errors between the perimeters of the inner n-gons (also taking $n = 4, 5, 8, 18, 36$ into account) and the perimeter of the circle.

Evaluation: Archimedes' method is theoretically sound and can be verified experimentally for relatively small values of n.

Estimation of π by Liu Hui

(a Chinese mathematician; publication in the year 263)

class of curves: circles (i.e., curves circumscribing a [today] known area of $S = \pi r^2$)

finite representation: inner regular n-gon

methodology: use areas S_n of these n-gons and the following formula (shown by Liu Hui for $n > 2$):

$$S_{2n} < S < S_{2n} + (S_{2n} - S_n)$$

example: $2n = 192$ (this is the maximum number as used by Liu Hui), then

$$\pi \approx 3{\cdot}1410 \quad \text{(true is } \pi = 3.14159265359\ldots\text{)}$$

(Liu Hui discussed the concept of "convergence" in these studies, which was not yet formulated by Archimedes.)

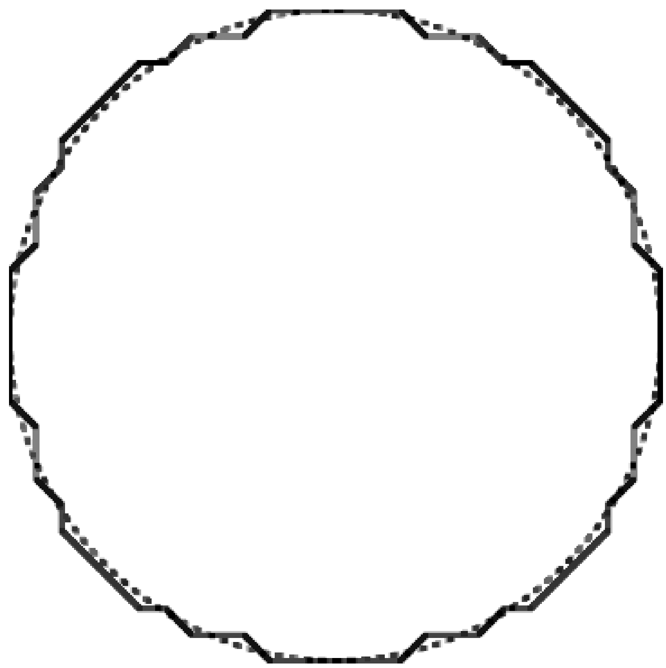
Evaluation: Liu Hui' method is theoretically sound and leads to an even faster convergence to the true value than Archimedes' method (but remains to be linear in convergence speed).

Grid-Intersection Digitization of Circles

class of curves: circle of diameter 1

finite representation: Ordered sequence of grid points defined by grid-intersection digitization in $\mathbb{G}_{nn}$ (i.e., a grid of size $n \times n$).

methodology: This ordered sequence is a simple polygon P_n with vertices at grid points and edges (between subsequent vertices) of length $1/(n-1)$ or $\sqrt{2}/(n-1)$. Use the perimeter $\mathcal{P}$ (i.e., sum of all edge lengths) of P_n for estimating π



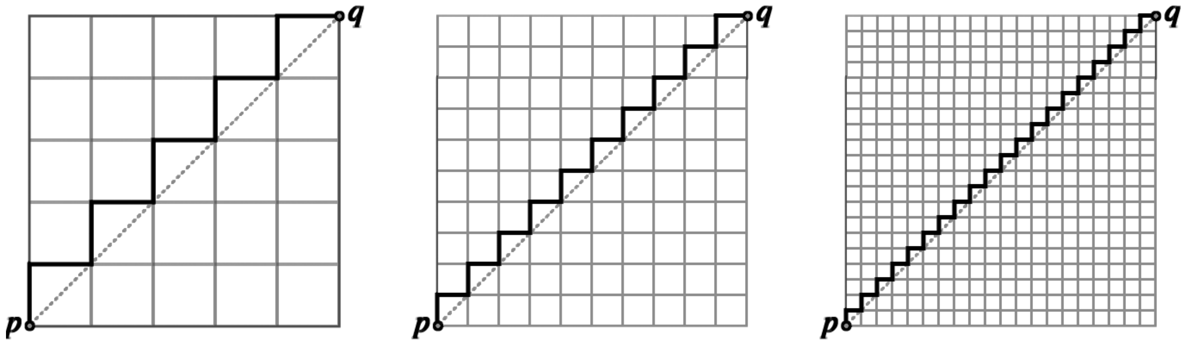
example: $\mathcal{P}(P_{26}) = 3.3439\ldots$ (i.e., an error of 6.4403%)

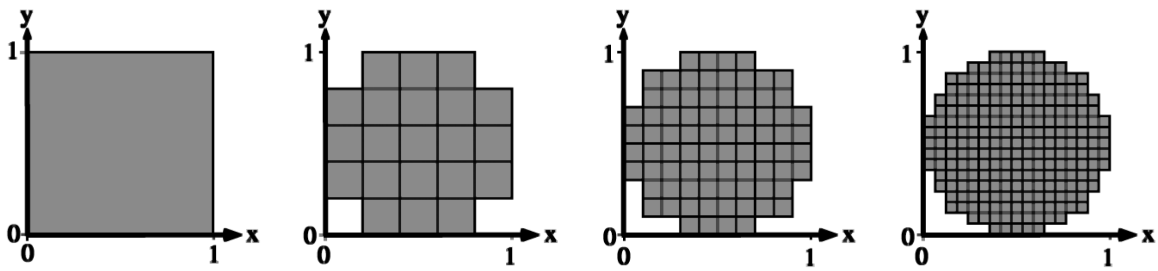
From a paper by M. Tajine and A. Daurat (2000) it follows that values $\mathcal{P}(P_n)$ do not converge to π, for n to ∞.

Evaluation: Theoretically not sound due to failing convergence, but errors might be acceptable if circle is contained in a "small square" (say, a square of at most 20 pixels on its edges).

The "Staircase Effect"

Assume a diagonal pq in a square with sides of length a. The length of the diagonal is equal to $a\sqrt{2}$. Now consider 4-path approximations $\rho(p, q)$ of the diagonal as shown below (i.e., for different grid resolutions). The length of these 4-paths is always equal to $2a$, whatever grid resolution will be chosen.





As a second example, consider the frontiers of digitized disks as shown above. Independent of grid resolution, the length of the frontier is always equal to 4.

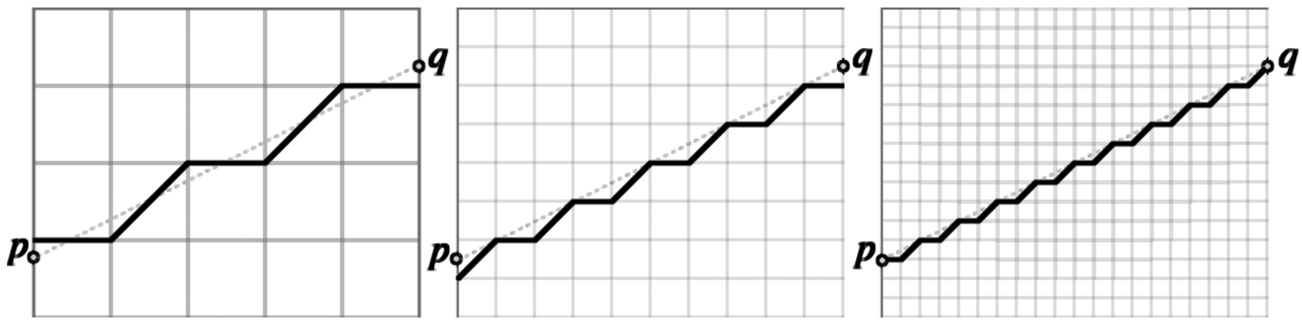
Evaluation: The use of the length of a 4-path for estimating the length of a digitized arc can lead to errors of $41.4214\ldots\%$, without any chance to reduce these errors in some cases by using higher grid resolution. This method cannot be used for length measurements in picture analysis.

Use of Weighted Edges

Now assume that we are using the length of an 8-path for length measurements (i.e., we use the *weight* $\sqrt{2}$ for diagonal edges, and just 1 as before for isothetic edges).

(A line or line segment in the Euclidean plane is *isothetic* iff it is parallel to one of the two Cartesian coordinate axes.)

We consider the line segment pq below with slope $22.5°$ and a length of $5\sqrt{5}/2$.



The length of $\rho(pq)$ is $3 + 2\sqrt{2}$ for grid constant 1 (shown on the left) and $(5 + 5\sqrt{2})/2$ for *all* grid constants $1/2^n$ ($n \geq 1$). This shows that the length of $\rho(pq)$ does not converge to $5\sqrt{5}/2$ as the grid constant goes to zero.

From a paper by M. Tajine and A. Daurat (2000) it follows that any length measurement based on weighted steps of 8-paths cannot lead to a length measurement which is convergent (to the true value) for increases in grid resolution.

Evaluation: Similar as for the use of 4-paths, but here only with errors of up to $7.9669\ldots\%$, without any chance to reduce these errors in some cases by using higher grid resolution.

Alternatives

Picture analysis is often directed (or based) on measuring properties such as the length of a curve, the contents of a region, the diameter of a set, the surface area of a volume, and so forth. Sound measurement methodologies are a fundamental requirement. These can be (for example, in the case of length measurements) based on:

(i) the use of optimized weights for local configurations of curves, where optimization is directed on curves of a specified class;

(ii) approximations of 4- or 8-curves by polygonal chains, where each edge of the chain is calculated based on global approximation constraints (e.g., segmenting an 8-path into subsequent DSSs of maximum length);

(iii) fitting of higher-order (e.g., second order) arcs to the given 4- or 8-curves; or

(iv) using stereological approaches (e.g., estimating length based on calculating intersection points with a finite set of straight lines).

In these lectures we will detail approach **(ii)** for length measurement.

Coursework

Related material in textbook: Sections 1.2.7 and 2.3.3.

A 5. [6 marks][a] Provide (hint: see Appendix)

(i) the missing formulas missing on page 5 for the case of outer polygons,

(ii) derive explicit formulas for the edges of the outer polygon;

(iii) and discuss conclusions for an approximative calculation of π (see the Appendix for an analogous discussion of the inner polygon).

Furthermore,

(iv) do the experiment indicated on page 8: calculate grid-intersection digitizations of circles in grids of varying grid resolution (defining digital circles having diameters of $30, 31, \ldots, 1000$ grid edges), and compare the length of the resulting 8-curves against the true perimeter of the circles. Show all the relative errors (i.e., absolute value of the difference between calculated length and true perimeter, divided by the true perimeter) in a diagram. (Hint: there is also a Bresenham algorithm for generating digital circles.)

[a] Two marks for the theoretical parts (i), (ii) and (iii), and four marks for part (iv).

Appendix: Archimedes' Iteration

We derive an explicit formula for edges e_m as used in our presentation of Archimedes' method.

triangle: $e_0 = r\sqrt{3} \approx r \cdot 1{\cdot}7321\ldots$

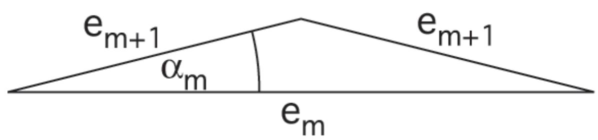
hexagon: $e_1 = \sqrt{2r^2 - r\sqrt{4r^2 - 3r^2}} = r$

12-gon: $e_2 = \sqrt{2r^2 - r\sqrt{4r^2 - r^2}} = r\sqrt{2 - \sqrt{3}} \approx r \cdot 0{\cdot}5176\ldots$

24-gon: $e_3 = r\sqrt{2 - \sqrt{2 + \sqrt{3}}} \approx r \cdot 0{\cdot}2611\ldots$

48-gon: $e_4 = r\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} \approx r \cdot 0{\cdot}1308\ldots$

96-gon: $e_5 = r\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} \approx r \cdot 0{\cdot}0654\ldots$



Note: In the repeated construction of $P_{n_{m+1}}$ from P_{n_m}, angle α_m goes to zero as $m \to \infty$, and e_{m+1} goes (from above) to $e_m/2$.

Let $m \geq 2$. Then we have $m - 2$ repeated stackings of $\sqrt{2} + \ldots$ in the formula for e_m.

Resulting examples of the perimeters:

triangle: $\mathcal{P}(P_3) = 3 \times 2^0 \times e_0 = 3r\sqrt{3} \approx 2r \cdot 2{\cdot}5982\ldots$

hexagon: $\mathcal{P}(P_6) = 3 \times 2^1 \times e_1 = 6r = 2r \cdot 3$

12-gon: $\mathcal{P}(P_{12}) = 3 \times 2^2 \times e_2 = 12r\sqrt{2 - \sqrt{3}} \approx 2r \cdot 3{\cdot}1056\ldots$

An Approximation Formula for π

Assume $r = 1$. In this case we have that $\mathcal{P}(P_{n_m}) - 2\pi$ goes to a constant (which is actually 2π) over n_m as $n_m \to \infty$.

This implies the following (note: here we have $m-2$ repeated stackings of $\sqrt{2}+\ldots$, assuming $m \geq 2$):

$$3 \times 2^{m-1}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots\sqrt{3}}}}} \to \pi$$

for $m \to \infty$. (This is just another approximation formula for π.)

Archimedes' Original Approach

In modern mathematical language, he used for initialization of his iteration at $m = 1$ (i.e, hexagon) the identities

$$\mathcal{P}(P_{n_m}) = 2n_m \sin(\pi/n_m) \quad \text{and} \quad \mathcal{P}(Q_{n_m}) = 2n_m \tan(\pi/n_m)$$

(note: $12\sin(\pi/6) = 6$ and $12\tan(\pi/6) = 4\sqrt{3}$, for $n_1 = 6$), and iterated according to the general recurrence formulae

$$\mathcal{P}(Q_{n_{m+1}}) = \frac{2 \cdot \mathcal{P}(P_{n_m}) \cdot \mathcal{P}(Q_{n_m})}{\mathcal{P}(P_{n_m}) + \mathcal{P}(Q_{n_m})}$$

$$\mathcal{P}(P_{n_{m+1}}) = \sqrt{\mathcal{P}(Q_{n_{m+1}}) \cdot \mathcal{P}(P_{n_m})}$$

(but without having these formulas explicitly at hand at his time; they are correct for $m \geq 0$). $n\sin(\pi/n)$ converges from below to π, for $n \to \infty$, and $n\tan(\pi/n)$ converges from above to π, for $n \to \infty$ (both with linear convergence speed).