Global properties of degree structures

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Overview

We will study structures from computability theory under the following aspects.

- Automorphism bases
- Biinterpretability with $\mathbb{N}$ in params, prime model question
- $\emptyset$-definable subsets

Automorphism bases

Definition 0.1 *A subset B of a structure $\boldsymbol{A}$ is an automorphism base*

$$\Leftrightarrow$$

the only automorphism of $\boldsymbol{A}$ fixing B pointwise is the identity.

Example. $B \subseteq (\mathbb{Q}, <)$ is a-base

$$\Leftrightarrow$$

B dense in $\mathbb{Q}$.

Structures based on $\leq_m$

- $\mathcal{D}_m$: all m-degrees
- $\mathcal{A}_m$: m-degrees of arithmetical sets
- $\mathcal{R}_m$: c.e. m-degrees

All three structures can be characterized as distributive usl's with a saturation property.

Theorem 0.2 *The minimal degrees do not form an a-base of $\mathcal{D}_m$, or of $\mathcal{A}_m$.*

Corollary 0.3 *$Aut(\mathcal{D}_m)$, $Aut(\mathcal{A}_m)$ are not simple*

Proof. $\{\pi \in \mathrm{Aut}(\boldsymbol{A}) : \pi$

fixes the minimal degrees$\}$ is a proper normal subgroup. $\diamondsuit$

A-bases of $\mathcal{R}_T$

Theorem 0.4 (Ambos-Spies) *For each* $\boldsymbol{c} \in \mathcal{R}_T$, $\boldsymbol{c} \neq \mathbf{o}$, $[\mathbf{o}, \boldsymbol{c}]$ *is an automorphism base.*

A modified proof of this result [N98] uses the following general method to show that B is an a-base of $\boldsymbol{A}$: provide a definable (relative to B)

$$H : \boldsymbol{A} \mapsto \tau B,$$

where τB is the collection of objects of type τ constructed from B.

τ-maps

Let $\widehat{\boldsymbol{x}} := [\mathbf{o}, \boldsymbol{x}]$. *Example of such a map:*

- $B = \widehat{\boldsymbol{c}}$
- $H(\boldsymbol{x}) = \{\widehat{\boldsymbol{y}} \cap \widehat{\boldsymbol{c}}, \widehat{\boldsymbol{z}} \cap \widehat{\boldsymbol{c}} : \boldsymbol{y} \vee \boldsymbol{z} = \boldsymbol{x}\}$.

In general:

- For each formula $\varphi(x)$, we have

$$H_\varphi(\boldsymbol{x}) = \{\boldsymbol{x} \in B : \varphi(\boldsymbol{x})\}$$

- If we have $H_1, \ldots, H_n$, for each fmla $\varphi(x, y_1, \ldots, y_n)$ have also $H(\boldsymbol{x}) = \{(H_1(\boldsymbol{y}_1), \ldots, H_n(\boldsymbol{y}_n)) : \varphi(\boldsymbol{x}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_n)\}$.

A-bases of $\mathcal{R}_m$

Theorem 0.5 *Each definable* $D \subseteq \mathcal{R}_m$, $D \not\subseteq \{\mathbf{0}, \mathbf{1}\}$ *forms an a-base of* $\mathcal{R}_m$.

Proof. Do it only for the set Min of minimal degrees in $\mathcal{R}_m$. Let

$\gamma(\boldsymbol{e}) \equiv \forall \boldsymbol{q}[\boldsymbol{e} \not\leq \boldsymbol{q} \Rightarrow \exists \boldsymbol{m} \in Min\,(\boldsymbol{m} \leq \boldsymbol{e} \;\&\; \boldsymbol{m} \not\leq \boldsymbol{q})]$

If $\widehat{\boldsymbol{e}} \cong \mathcal{R}_m$ effectively, then $\gamma(\boldsymbol{e})$. Let

$$H(\boldsymbol{x}) = \{[\mathbf{0}, \boldsymbol{y}] \cap Min : \boldsymbol{y} \geq \boldsymbol{x} \;\&\; \gamma(\boldsymbol{y})\}$$

- If $\boldsymbol{z} \not\leq \boldsymbol{x}$, then using Denisov's characterization, there is $\boldsymbol{y} \geq \boldsymbol{x}, \boldsymbol{y} \not\geq \boldsymbol{z}$ such that $\gamma(\boldsymbol{y})$.
- Then, if $H(\boldsymbol{z}) = H(\boldsymbol{x})$, there must be an $\boldsymbol{u} \geq \boldsymbol{z}$, $\gamma(\boldsymbol{u})$, s.t. $[\mathbf{o}, \boldsymbol{u}] \cap Min = [\mathbf{o}, \boldsymbol{y}] \cap Min$, contradiction.

Question 1 *What is the structure of $Aut(\mathcal{R}_m)$? Is it a simple group?*

From $\mathcal{R}_m - \{\mathbf{1}\} \cong$ the Δ^0_2 m-degrees (Denisov), we can infer the existence of an automorphism of order 2.

$\mathcal{R}_T$

The property $\gamma(\boldsymbol{e})$ plays a role similar to Ambos-Spies' "downward splitting property": let DSP($\boldsymbol{a}$) $\Leftrightarrow$

$$\forall b \not\leq \boldsymbol{a}\ \forall \boldsymbol{c} \not\leq \boldsymbol{b} \vee \boldsymbol{a}[\widehat{\boldsymbol{c}} \cap \widehat{\boldsymbol{a}} \neq \widehat{\boldsymbol{c}} \cap \widehat{\boldsymbol{b}}].$$

To prove $\widehat{\boldsymbol{c}}$ is an a-base, use as $H(\boldsymbol{x})$ a complex object built up from ideals $\widehat{\boldsymbol{c}} \cap \widehat{\boldsymbol{a}}$, where $DSP(\boldsymbol{a})$: for $\boldsymbol{x} \in NC$, let $H(\boldsymbol{x}) =$

$$\{\{\ \langle\{\widehat{\boldsymbol{c}}\cap\widehat{\boldsymbol{a}} : \boldsymbol{a} \in \widehat{\boldsymbol{p}}_0\cap\mathrm{DSP}\}, \{\widehat{\boldsymbol{c}}\cap\widehat{a} : \boldsymbol{a} \in \widehat{\boldsymbol{p}}_1\cap\mathrm{DSP}\}\rangle :$$

$$\boldsymbol{p}_0 \vee \boldsymbol{p}_1 = \boldsymbol{p}, \boldsymbol{p}_i \in \mathrm{NC}\},$$

$$\{\ \langle\{\widehat{\boldsymbol{c}}\cap\widehat{\boldsymbol{a}} : \boldsymbol{a} \in \widehat{\boldsymbol{q}}_0\cap\mathrm{DSP}\}, \{\widehat{\boldsymbol{c}}\cap\widehat{a} : \boldsymbol{a} \in \widehat{\boldsymbol{q}}_1\cap\mathrm{DSP}\}\rangle :$$

$$\boldsymbol{q}_0 \vee \boldsymbol{q}_1 = \boldsymbol{q}, \boldsymbol{q}_i \in \mathrm{NC}\} :$$

$$\boldsymbol{p} \vee \boldsymbol{q} = \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{q} \in \mathrm{NC}\}$$

A-bases of $\mathcal{E}^*$

Theorem 0.6 *Max*$=\{M^* : M$ *maximal*$\}$ *is an a-base.*

Proof. $H(x) =$

$\{[y_0, 1] \cap \mathrm{Max}^*, [y_1, 1] \cap \mathrm{Max}^* : y_0 \vee y_1 = x\}.$ $\diamondsuit$

Since each maximal element of $\mathcal{E}^*$ is determined by its recursive subsets, it follows that Rec* is an a-base.

Also the creative sets are (each infinite c.e. set is the union of two creative ones).

Question 2 *Is each orbit of* $\mathcal{E}^*$ *an a-base ?*

Part II: BI with $\mathbb{N}$ in params

Definition 0.7 *A copy of* $\mathbb{N}$ *coded in* $\boldsymbol{A}$ *with params is a structure*

$$\boldsymbol{M} = (D, R_+, R_\times) \cong \mathbb{N}$$

such that D *and the relations* $R_+, R_\times$ *are param-definable in* $\boldsymbol{A}$*.*

Use the letter $\boldsymbol{M}$ for such coded structures.

Definition 0.8 $\boldsymbol{A}$ *is biinterpretable with* $\mathbb{N}$ *in params*

$$\Leftrightarrow$$

$\exists M \exists f : A \mapsto M[f\ 1-1, \textit{param-def}]$.

Known for $\mathcal{D}(\leq \emptyset')$ (Slaman/Woodin).

Consequences

- Can define a copy $N(\boldsymbol{A})$ of $\mathbb{N}$ without params
- $\boldsymbol{A}$ is a prime model and and a minimal model of $\mathrm{Th}(\boldsymbol{A})$. In fact, for $\boldsymbol{B}, \boldsymbol{C} \equiv \boldsymbol{A}$,

$$\boldsymbol{B} \prec \boldsymbol{C} \;\Leftrightarrow\; N(\boldsymbol{B}) \prec N(\boldsymbol{C}).$$

- Arithmetical $\Leftrightarrow$ param-definable
- Each $\pi \in \mathrm{Aut}(\boldsymbol{A})$ is arithmetical. Finite a-base.

(Recall: $\boldsymbol{A}$ prime $\Leftrightarrow$ each n-orbit definable, and $\boldsymbol{A}$ minimal $\Leftrightarrow$ there is no proper elementary submodel.)

$\mathcal{R}_m$

BI fails for $\mathcal{R}_m$. In fact,

- $|\mathrm{Aut}(\mathcal{R}_m)| = 2^\omega$
- $\exists \boldsymbol{e}[\mathbf{0}, \boldsymbol{e}) \prec \mathcal{R}_m - \{\mathbf{1}\}]$

Question 3 *Is $\mathcal{R}_m$ prime ?*

Theorem 0.9 *$\mathcal{R}_m$ prime $\Rightarrow \exists k$ [each realized type is principal via a Σ^0_k-formula]*

Theorem 0.10 *The arithmetical m-degrees form a prime model.*

The BI-conjecture for $\mathcal{R}_T$

Question 4 *Is $\mathcal{R}_T$ biinterpretable with $\mathbb{N}$ in params ?*

Weaker:

Question 5 *Is $\mathcal{R}_T$ prime ? ω-homogeneous ? Minimal ?*

An approx to BI without params

Theorem 0.11 (with Shore, Slaman) *There is a $\emptyset$-definable*

$$f : \mathcal{R}_T \mapsto N(\mathcal{R}_T)$$

such that

$$\boldsymbol{x}^{(2)} \neq \boldsymbol{y}^{(2)} \Rightarrow f(\boldsymbol{x}) \neq f(\boldsymbol{y}).$$

Using a-bases

Suppose $B \subseteq \mathcal{R}_T$ is param-definable, $H : \mathcal{R}_T \mapsto \tau B$ is 1-1, definable. Then to show BI it suffices to produce M and a 1-1 param-definable $g : B \mapsto M$:

$$\mathcal{R}_T \overset{H}{\mapsto} \tau B \overset{\tau g}{\mapsto} \tau \boldsymbol{M} \mapsto \boldsymbol{M}.$$

Even if no H is known, we still obtain a finite a-base: the params for M, g.

Separating sequences

Definition 0.12 *A sequence* $(\boldsymbol{g}_i)_{i\in\omega}$ *separates* B *from*

- *below* $\Leftrightarrow$ $\forall \boldsymbol{x}, \boldsymbol{y} \in B$

  $\boldsymbol{x} \not\leq \boldsymbol{y} \Rightarrow \exists i(\boldsymbol{g}_i \leq \boldsymbol{x}, \boldsymbol{g}_i \not\leq \boldsymbol{y})$,

- *from above* $\Leftrightarrow \forall \boldsymbol{x}, \boldsymbol{y} \in B$

  $\boldsymbol{x} \not\leq \boldsymbol{y} \Rightarrow \exists i(\boldsymbol{x} \not\leq \boldsymbol{g}_i, \boldsymbol{y} \leq \boldsymbol{g}_i)$.

$\boldsymbol{M}$ separates B if the sequences $(i^{\boldsymbol{M}})_{i\in\omega}$ does. In this case, obtain a 1-1 param-definable $g : B \mapsto M$: e.g. for downward sep. $\boldsymbol{M}$,

$g(\boldsymbol{x})$ = arithm. index for

$\{i : i^{\boldsymbol{M}} \leq \boldsymbol{x}\}$.

Final segments

Theorem 0.13 (Shore, Slaman)

- *There is a finite injury construction for* $\boldsymbol{M}$, $\boldsymbol{b} < \boldsymbol{1}$ *s.t.* $\boldsymbol{M}$ *separates* $[\boldsymbol{b}, \boldsymbol{1}]$.
- $\boldsymbol{M}$ *exists for each promptly simple degree* $\boldsymbol{b}$.

It follows that the finite sets of ps degrees are unif'ly definable. The domain of $\boldsymbol{M}$ is a SW-set:

$M = \{\boldsymbol{x} \leq \boldsymbol{c} \text{ minimal} :$

$\quad \boldsymbol{q} \leq \boldsymbol{x} \vee \boldsymbol{p}\}.$

Question 6 *Is there a nontrivial final segment of* $\mathcal{R}_T$ *which forms an a-base ?*

Question 7 *Is there a u.c.e. antichain which forms an a-base ?*

Definable antichains

Other ways to define antichains:

- $\{\boldsymbol{x} \leq \boldsymbol{c} \text{ maximal} : \ \boldsymbol{q} \not\leq \boldsymbol{x} \vee \boldsymbol{p}\}$
  (Harrington,Shelah; Harrington, Slaman)
- maximum $\boldsymbol{a}$-cappable; only known to yield arbitrarily large finite a-chains (A-Sp, Shore, Hirschfeld) but works in all intervals
- $\{\boldsymbol{x} \leq \boldsymbol{c}, \boldsymbol{d} \text{ max} :$

  $\boldsymbol{x}$ not top of a diamond$\}$

  (A-Sp, Soare; Lempp,N)

Initial segments

- Have to separate upward
- Problem: the constructions yield u.c.e. antichains.

Proposition 0.14 *If* $(\boldsymbol{g}_i)$ *is a u.c.e. sequence of nonzero degrees and* $\boldsymbol{c} \neq \mathbf{o}$, *then*

$$\exists \boldsymbol{v} < \boldsymbol{u} \leq \boldsymbol{c}\ \forall i [\boldsymbol{u} \vee \boldsymbol{g}_i = \boldsymbol{v} \vee \boldsymbol{g}_i].$$

Possible solutions:

- Find a more flexible construction of a definable a-chain
- use different types of interaction of $\boldsymbol{M}$ and $[\mathbf{o}, \boldsymbol{c}]$
- forget it.

Approximations

Theorem 0.15 (N, ?) *There is*

- $\boldsymbol{c} \neq \mathbf{o}$
- $X \subseteq \widehat{\boldsymbol{c}}$
- $f : X \mapsto \boldsymbol{M}$ *1-1*

such that

$$\forall \boldsymbol{u} \leq \boldsymbol{c}, \boldsymbol{u} \neq \mathbf{o} \exists \boldsymbol{x} \in X$$

$$[\boldsymbol{x} \neq \mathbf{o} \;\&\; \boldsymbol{x} \leq \boldsymbol{u}]$$

Proof.

- Construct c.e. sets C, D
- Let $X =$ degrees of c.e. set splits of C
- Construct $\boldsymbol{M}$ separating X from above. Domain of $\boldsymbol{M}$ is a SW set.

The advantage of set splits U: control size of U changes in reaction to C changes.

If, in addition, we make C nonbounding, we obtain a strictly descending definable sequence $(\boldsymbol{b}_i)$ s.t.

$\forall \boldsymbol{x} \leq \boldsymbol{c}[\boldsymbol{x} \neq \mathbf{o} \Rightarrow \exists i\ \boldsymbol{b}_i \leq \boldsymbol{x}]$.

Part III: $\emptyset$-definability

Examples of interesting $\emptyset$-definable subsets of $\mathcal{R}_T$:

- Promptly simple degrees (A-Spies e.a.)
- $\boldsymbol{L}_n, \boldsymbol{H}_{n-1}$ for $n \geq 2$ (N, Shore, Slaman)
- Contiguous degrees (Downey, Lempp)

Question 8 *Is there a $\emptyset$-definable ideal except the cappable degrees and the noncuppable degrees ?*

Weaker, but still interesting: proper ideals generated by a definable set.

Theorem 0.16 *The ideal generated by NB is properly contained in CAP.*

Local structure of d'ble sets

Question 9 (Li-Ansheng) *Is there a $\emptyset$-definable D and an interval $\boldsymbol{a}, \boldsymbol{b}]$ such that $\boldsymbol{a}, \boldsymbol{b} \notin D$ and $|[\boldsymbol{a}, \boldsymbol{b}] \cap D| = 1$?*

Good candidates for D: contiguous; sup of a minimal pair. *Note: probably solved by Cholak,Downey and Walk*

$\emptyset$-d'ty of a copy of $\mathbb{N}$

Can we define a copy $N(\boldsymbol{A})$ of $(\mathbb{N}, +, \times)$ in $\boldsymbol{A}$ without params ?

- Possible for $\mathcal{R}_m$ [N95], $\mathcal{R}_T$ [N,Shore, Slaman 96] and for $\mathcal{R}_{wtt}$ [Nta].
- Fails for $\mathcal{E}^*$.
- Unknown for $\mathcal{R}_Q$, $\mathcal{R}_{tt}$