Coding and Definability in $\mathcal{R}_T$

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Coding and theories

Consider a structure $\mathbf{A}$ from computability theory (such as $\mathcal{D}_T$, $\mathcal{R}_T$ or $\mathcal{E}$). Coding methods can be used to show that $Th(\mathbf{A})$ is complex.

1. Uniform coding of a sufficiently complicated class of structures with parameters, shows $Th(\mathbf{A})$ undecidable.
2. Coding of a standard model of arithmetic with parameters, implies $K \leq_m Th(\mathbf{A})$.

So far, the coding is *local*, namely reflects properties of very special parameter lists in the context of $\mathbf{A}$.

3. Extend (2) to give an interpretation of true arithmetic (or, true second–order arithmetic) in $\mathbf{A}$.

To do so, need a first–order condition on parameters coding models which imply standardness.

- Use some uniform definability result (like an exact pair theorem), or
- compare coded models, typically by considering definable (partial) maps between them.

What is coding ?

Example:
Each finite distributive lattice is isomorphic to an initial segment $[\mathbf{0}, \mathbf{a}]$ of $\mathcal{R}_m$.

A coding (or interpretation) of a relational structure $\mathbf{B}$ in $\mathbf{A}$ is given by a scheme of formulas

$$\phi_U(x,\overline{p}),(\phi_R(x,\overline{p}))_{R \text{ relation symbol}})$$

s.t. for appropriate $\overline{p}$ in $\mathbf{A}$, a copy of $\mathbf{B}$ is defined on $\{x : \mathbf{A} \models \phi_U(x,\overline{p})\}$.

A more general coding: $\mathbb{Q}$ is coded in $\mathbb{Z}$ via the quotient field construction

Coding and structure

Coding methods can reveal information about the structure $\mathbf{A}$ itself (not only it's theory), such as

- restrictions on automorphisms (e.g. for $\mathcal{D}_T$)
- show that $\mathbf{A}$ is a prime model of its theory (Slaman–Woodin for $\mathcal{D}_T(\leq 0'))$
- compare the complexity of models (e.g. $\mathcal{E}$ can be interpreted in $\mathcal{R}_T$, but not conversely)
- definability results.

Invariance and definability

Suppose $C \subseteq \mathbf{A}^n$ is given "externally". When is C already inherent in $\mathbf{A}$?

- weak version: C is invariant under automorphisms
- strong: C is first–order definable in $\mathbf{A}$ without parameters.

Examples:

- The jump operator is definable in $\mathcal{D}_T$ [Cooper ta]
- In $\mathcal{R}_T$, promptly simple=noncappable.

Both first order definitions are using simple, "understandable" formulas.

Definability in $\mathcal{R}_T$

We prove definability results using coding methods.

Theorem 0.1 $\{\mathbf{x} : \mathbf{x}^{(2)} = \mathbf{a}^{(2)}\}$ *is definable for each* $\mathbf{a}$.

Thus Low_2 and $High_2$ are definable.

Definition 0.2 *We write* $\mathbf{x} \sim_2 \mathbf{y}$ *if* $\mathbf{x}^{(2)} = \mathbf{y}^{(2)}$.

Theorem 0.3 (Main definability Theorem) *If*

- $C \subseteq \mathcal{R}_T^n$ *is invariant under* $\sim_2$ *and*

- *the corresponding relation on indices is arithmetical,*

then C is definable.

Applications: definable are

- Low_n, $High_n$ for each $n \geq 2$
- The relation $\sim_2$ itself
- For each $n \geq 2$, the relation " $\mathbf{x}^{(n)} \leq \mathbf{y}^{(n)}$".

$High_1$

Theorem 0.4 *$High_1$ is definable*

$$\mathbf{x} \in High_1 \Leftrightarrow \mathcal{R}_T \models (\forall \mathbf{y}(\exists \mathbf{z} \leq \mathbf{x})[\mathbf{z} \sim_2 \mathbf{y}].$$

For "$\Rightarrow$", use two times the Robinson jump interpolation in relativized forms.
For the direction $\Leftarrow$, use a result of Soare and Stob (relativized to $0^{(2)}$):
If $\mathbf{c} \in \mathcal{R}_T - \{0\}$, then there is a $\mathbf{u}$ r.e.a. $\mathbf{c}$, which is not an r.e. degree.

First goal: invariance

Lemma 0.5 *If $\mathbf{a}, \mathbf{b}$ are automorphic, then $\mathbf{a}^{(2)} = \mathbf{b}^{(2)}$*

Proof. $\mathbf{a}^{(2)}$ is determined by the class of sets $\Sigma^0_3(\mathbf{a})$. We will recover this class from the isomorphism type of $(\mathcal{R}_T, \mathbf{a})$.
We define a class of sets $S(\mathbf{a})$ which only depends on this isomorphism type, and show $S(\mathbf{a}) = \Sigma^0_3(\mathbf{a})$. Then:
$\mathbf{a}, \mathbf{b}$ automorphic
$\Rightarrow S(\mathbf{a}) = S(\mathbf{b})$
$\Leftrightarrow \Sigma^0_3(\mathbf{a}) = \Sigma^0_3(\mathbf{b})$
$\Leftrightarrow \mathbf{a}^{(2)} = \mathbf{b}^{(2}$.

Ultimately we can recover $\Sigma^0_3(\mathbf{a})$ because, on $[0, \mathbf{a}]$, $\leq_T$ is $\Sigma^0_3(\mathbf{a})$ and join is effective.

Towards defining $S(\mathbf{a})$
First, we need a representation of sets$X \subseteq \omega$ in $\mathcal{R}_T$. We code the model (ω, σ, X), where σ is the successor function.
To keep in mind:
Upper bound–don't represent too many sets
Lower bound–represent all sets in $\Sigma^0_3(\mathbf{a})$.

The coding frame

Definition 0.6 (The domain) *A set $G \subseteq \mathbf{R}_T$ is called an SW*-set *if, for some parameters $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{b} \in \mathcal{R}_T$, G is the set of minimal degrees $\mathbf{x} \in [\mathbf{b}, \mathbf{r}]$ satisfying $\mathbf{q} \leq \mathbf{x} \vee \mathbf{p}$.*

Definition 0.7 (Successor) *An* effective successor model *is a standard successor structure $\mathbf{G} = (\{\mathbf{g_i} : \mathbf{i} \in \omega\}, \sigma)$ coded by parameters $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{b}$ and four further parameters $\mathbf{e}_0, \mathbf{f}_0, \mathbf{e}_1, \mathbf{f}_1$ as follows:*

- *$\{\mathbf{g}_i : i \in \omega\}$ is a SW-set*
- *For each i,*
  - $(\mathbf{g}_{2i} \vee \mathbf{e}_1) \wedge \mathbf{f}_1 = \mathbf{g}_{2i+1}$
  - $(\mathbf{g}_{2i+1} \vee \mathbf{e}_0) \wedge \mathbf{f}_0 = \mathbf{g}_{2i+2}$
  - $\mathbf{g}_{2i} \not\leq \mathbf{f}_1$ *and* $\mathbf{g}_{2i+1} \not\leq \mathbf{f}_0$

Effectivity

Lemma 0.8 *If there is a low$_2$ upper bound on all parameters, then, for some $\beta \leq \emptyset''$,*

$$\mathbf{g}_i = deg_T(\{\beta(i)\}^R.$$

Here R is some set in the degree $\mathbf{r}$.
The function β is defined by recursion. E.g.,

$$\beta(2n+1) = \text{some } e \text{ s.t. } Z = \{e\}^R \text{ is total,}$$
$$\text{and } Z \leq E_1 \oplus \{\beta(2n)\}^R, F_1.$$

Definition 0.9 *Given $\mathbf{a}$ and an upper bound on parameters $\mathbf{u}$. $X \subseteq \omega$ is* represented below $\mathbf{a}, \mathbf{u}$ *if*

- *there are parameters below $\mathbf{u}$ coding effective successor model*
- *$\mathbf{r} \leq \mathbf{a}$ (and hence all $\mathbf{g}_i \leq \mathbf{a}$)*
- *there are also $\mathbf{c}, \mathbf{d} \leq \mathbf{a}$ such that*

$$X = \{i : \mathbf{c} \leq \mathbf{g}_i \vee \mathbf{d}\}$$

$$S(\mathbf{a})$$

Let

$$S(\mathbf{a}) = \{X : \forall \mathbf{u} \text{ noncappable } \}$$

$$X \text{ represented below } \mathbf{a}, \mathbf{u}\}$$

FACT: If $\mathbf{a}, \mathbf{b}$ are automorphic, then $S(\mathbf{a}) = S(\mathbf{b})$.
(Reason: If π is an automorphism such that $\pi(\mathbf{a}) = \mathbf{b}$, then π maps representations of X below $\mathbf{a}, \mathbf{u}$ to representations of X below $\mathbf{b}, \pi(\mathbf{u})$).

$$S(\mathbf{a}) = \Sigma^0_3(\mathbf{a})$$

This is the crucial step from an arithmetical property of $\mathbf{a}$ to an invariant property of $\mathbf{a}$.
Proof: 1. The upper bound
$S(\mathbf{a}) \subseteq \Sigma^0_3(\mathbf{a})$ uses the effectivity of the coding. Choose $\mathbf{u}$ low noncappable. Then, for a representation below $\mathbf{a}, \mathbf{u}$

$$i \in X \Leftrightarrow C \leq_T \{\beta(i)\}^R \oplus D$$

and the right hand side is $\Sigma^0_3(\mathbf{a})$.
2. The lower bound
$\Sigma^0_3(\mathbf{a}) \subseteq S(\mathbf{a})$ requires more work.
Given: X in $\Sigma^0_3(\mathbf{a})$, $\mathbf{u}$ promptly simple.
Find: $\mathbf{G}$ coded below $\mathbf{a}, \mathbf{u}$, and $\mathbf{c}, \mathbf{d} \leq \mathbf{a}$ such that

$$X = \{i : \mathbf{c} \leq \mathbf{g}_i \vee \mathbf{d}\}.$$

Big Theorems

Theorem A: If: $\mathbf{a} \neq 0$, $\mathbf{u}$ promptly simple, then:
there is an effective successor model coded below $\mathbf{a}, \mathbf{u}$ such that

- $(\mathbf{g}_i)$ is u.r.e. and
- $\mathbf{r} = \bigoplus_i \mathbf{g}_i$ is low.

Theorem B: If

- $(\mathbf{g}_i)$ is a u.r.e. antichain,
- $\bigoplus_i \mathbf{g}_i$ is low and
- $\mathbf{g}_i < \mathbf{a}$ for each i, then:

for each X in $\Sigma^0_3(\mathbf{a})$, there are $\mathbf{c}, \mathbf{d} \leq \mathbf{a}$ such that

$$X = \{i : \mathbf{c} \leq \mathbf{g}_i \vee \mathbf{d}\}.$$

Relativizations of $\mathcal{R}_T$

Let $\mathcal{R}^X_T = \{\mathbf{y} : \mathbf{y} r.e.a. X\}$.

Theorem 0.10 *If $Z^{(2)} \neq W^{(2)}$, then $\mathcal{R}_T^Z \not\cong \mathcal{R}_T^W$*

(Similar Theorems are known for $\mathcal{D}_T(\leq 0')$, $\mathcal{R}_m$ and $\mathcal{E}$.)
Proof: By relativization of the arguments above, also in $\mathcal{R}_T^X$, $S(\mathbf{a}) = \Sigma_3^0(\mathbf{a})$.
So, from the isomorphism type of $\mathcal{R}_T^X$, we can recover

$$\Sigma_3^0(X) = \bigcap\{S(\mathbf{a}) : \mathbf{a} \text{ r.e.a in } X, \mathbf{a} \neq 0\}$$

Definability of $\{\mathbf{x} : \mathbf{x} \sim_2 \mathbf{a}\}$

To define $\{\mathbf{x} : S(\mathbf{x}) = \Sigma_3^0(\mathbf{a})\}$ by a f.o. formula, we need

- a coding of SMA's $\mathbf{M}$, to evaluate "$\Sigma_3^0(\mathbf{a})$" inside $\mathcal{R}_T$
- a way to f.o.-define isomorphism between SMA's, to express standardness and compare representations in different models
- A way to f.o.–define the isomorphism between an effective successor model $\mathbf{G}$ and $\mathbf{M}$ (viewed as successor model), to transfer a represented set $X \subseteq \mathbf{G}$ into $\mathbf{M}$.

Schemes

Recall: a *scheme* is a collection of formulas (with parameters) to code objects of a certain type in $\mathcal{R}_T$. From now on it may include a f.o.–*correctness condition* on parameters.
Example: We obtained s_G, a scheme for coding effective successor models. The c.c. says it is a successor model.
Convention: If s_X is a scheme, X,X_0 etc denote objects coded via this scheme, by parameters satisfying the correctness condition.

Standard models and comparison

Theorem 0.11 *1. There is a scheme s_M such that all coded $\mathbf{M}$ are standard models of arithmetic.*

2. There is a scheme s_h for uniformly defining the isomorphism between any two standard $\mathbf{M}$'s,

3. There is a scheme s_g for uniformly defining the isomorphism between standard $\mathbf{G}$'s such that (G_i) is u.r.e. and the parameter $\mathbf{r}$ is low and any $\mathbf{M}$'s, viewed as a successor model.

- These restricted $\mathbf{G}$'s were the only models we needed to verify that $\Sigma_3^0(\mathbf{a}) \subseteq S(\mathbf{a})$.
- So we can add a correctness condition on the scheme s_G stating that an isomorphism $\tilde{g}$ to some standard $\mathbf{M}_G$ exists.

- This "secret" modification in the def of representability below $\mathbf{a}, \mathbf{u}$ and hence in the definition of $S(\mathbf{a})$ doesn't change the fact that

$$\Sigma^0_3(\mathbf{a}) = S(\mathbf{a})$$

.

To prove the theorem, use the following easy extension of the SW-construction.

Lemma 0.12 *If $(\mathbf{u}_i)$ is a u.r.e. sequence, and $\bigoplus_i \mathbf{u}_i$ is low , then there is $\mathbf{M}$ such that $i^{\mathbf{M}} \leq \mathbf{u}_i$ and*

$$\mathbf{u}_i \not\leq \mathbf{u}_j \Rightarrow i^{\mathbf{M}} \not\leq \mathbf{u}_j.$$

A version of $S(\mathbf{x})$ inside $\mathbf{M}$

Let $(T_j) = (W_j^{\emptyset^3})$ be a listing of the Σ^0_4 sets.

$S(\mathbf{x}, \mathbf{M}) = \{j \in \mathbf{M} : \forall \mathbf{u}$ noncappable
$\exists \mathbf{G}$ coded below $\mathbf{x}, \mathbf{u}$
$\exists g : \mathbf{G} \leftrightarrow \mathbf{M}$ isom. of successor models

$$g^{-1}(T_j) \text{ is repr. below } \mathbf{x}, \mathbf{u} \text{ via } \mathbf{G}$$

Then $\mathbf{x} \sim_2 \mathbf{a} \Leftrightarrow \exists \mathbf{M} \forall j \in \mathbf{M}$

$$[j \in S(\mathbf{x}, \mathbf{M}) \Leftrightarrow \text{“}\mathbf{M} \models T_j \in \Sigma^0_3(\mathbf{a})\text{”}],$$

and the statement on the right can be expressed in a f.o.–way. by quantifying over the parameters involved and using that each $\mathbf{M}$ is standard.

Approximation to biinterpretability

Theorem 0.13 *(i) A SMA $\mathbf{N}$ can be interpreted in $\mathcal{R}_T$ without parameters.*
(ii) There is a definable map $f : \mathbf{R}_T \to \mathbf{N}$ such that $(\forall \mathbf{a}) W''_{f(\mathbf{a})} = \mathbf{a}''$.

Part (ii) gives the Main Definability Theorem: If

- $C \subseteq \mathcal{R}^n_T$ is invariant under $\sim_2$ and
- the corresponding relation on indices is arithmetical,

then C is definable.
Proof: $C = f^{-1}(f(C))$.