Borel structures and Borel theories

André Nies

Jan 19, 2012

Presentations of continuum size structures

Structures of size the continuum occur naturally in analysis, algebra, and other areas.
Examples are the additive groups of real numbers, or of p-adic integers, and the ring of continuous functions on the unit interval.

What can we say about the complexity of these structures?

- We will study continuum size structures that are effective in the weak sense that they have a Borel presentation.
- This includes the examples given above.
- It actually includes most continuum-sized structures from mathematics.

The plan

- We give some background from descriptive set theory
- We define Borel presentations of structures. We give examples.
- We show that there is a Borel (even Büchi) structure without an injective Borel presentation.
- We consider the interplay of Borel theories and Borel structures, and show that the completeness theorem from logic fails in the Borel setting for an uncountable signature.
- We end with open questions.

Basic concepts

- Let $\mathcal{X}$ be a "standard Borel space", for instance $\mathbb{R}$, Cantor space $\{0,1\}^{\omega}$, or, more generally, some uncountable Polish topological space (that is, complete metrizable, and separable).
- The Borel sets of $\mathcal{X}$ are the smallest σ-algebra containing the open sets.
- Thus, the Borel sets are the sets obtained from the open sets by iterated applications of complementation and countable union.

Sets that are Borel, and sets that are not Borel

- Many sets of reals one from analysis are Borel. For instance, if $f: [0, 1] \to \mathbb{R}$ is an *arbitrary* function, the set $\{z: f'(z) \text{ exists}\}$ is Borel. This can be proved using the Denjoy-Young-Saks Theorem.
- However, using the axiom of choice, one can define a set that is not Borel: for instance, a separating set for the Vitaly equivalence relation on $\mathbb{R}$.
- The set of wellorderings on ω is not Borel.
- An ultrafilter on ω can be viewed as a subset of Cantor space 2^ω. If the ultrafilter is free, it is not Borel; not even measurable.

Larger classes of sets

$\boldsymbol{\Sigma}^1_1$ (or analytic)	projections of Borel relations
$\boldsymbol{\Pi}^1_1$ (or co-analytic)	complements of $\boldsymbol{\Sigma}^1_1$ sets

Souslin (1917) proved that

$$\text{Borel} = \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1.$$

We will use analogy with computable structures

Computable model theory studies the effective content of model theoretic concepts and results for countable structures.

Definition

A structure $\mathbf{A}$ in a finite signature is called computable if

- there is a function from a computable set $D \subseteq \omega$ onto $\mathbf{A}$, such that
- the preimages of the atomic relations of $\mathbf{A}$ (including equality) are computable.

Example

$(\mathbb{Q}, <)$ and $(\omega, +, \times)$ are computable.

We replace “computable” by “Borel”

Main Definition (H. Friedman, 1979)

A structure $\mathbf{A}$ in a countable signature is called Borel structure if

- there is a function from a Borel set D in a standard Borel space $\mathcal{X}$ onto $\mathbf{A}$, such that
- the preimages of the atomic relations of $\mathbf{A}$ (including equality) are Borel relations on $\mathcal{X}$.

We call D together with these preimages of the atomic relations a Borel presentation of $\mathbf{A}$.

- We allow that an element of $\mathbf{A}$ is represented by a whole equivalence class of a Borel equivalence relation E.
- If E is the identity relation on D, we say the Borel presentation is injective.

Examples of Borel structures in algebra

Many algebraic structures of size the continuum are Borel. (The signature is finite.)

- The Boolean algebra $\mathcal{P}(\omega)/\text{Fin}$ is Borel, where Fin denotes the ideal of finite sets. More generally, consider BAs of the form $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I} \supseteq \text{Fin}$ is a Borel ideal. This gives lots of non-isomorphic examples.

- For a countable structure S in a countable signature, the following structures are Borel:
  - the lattice of substructures of S,
  - the congruence lattice of S,
  - the automorphism group of S.

These structures are in fact arithmetical relative to the atomic diagram of S.

Borel structures in analysis

Most structures from analysis of size the continuum are Borel.
For instance, the fields $(\mathbb{R}, +, \times)$, $(\mathbb{C}, +, \times)$, the ring $\mathcal{C}[0, 1]$ are Borel.

Fact (Hjorth and Nies, 2010)

There are continuum many non-Borel isomorphic injective Borel presentations of $(\mathbb{R}, +)$.

Proof. For each $p > 1$, we obtain a Borel presentation of $(\mathbb{R}, +)$ as the abelian group underlying the Banach space

$$\ell_p = \{\vec{x} \in \mathbb{R}^\omega : \sum_n |x_n|^p < \infty\},$$

where the norm is $\|\vec{x}\|_p = (\sum_n |x_n|^p)^{1/p}$.
Two different Borel presentations of this sort are not Borel isomorphic, because any two Borel isomorphic Polish groups are already homeomorphic. (See Thm. 9.10 in Kechris, Classical Descriptive set Theory.)

Restrictions on Borel orders

- The well-ordering of length $2^{\aleph_0}$ is **not** Borel.
  (If $\mathcal{S}$ is a Borel presentation of this well-ordering, then the class $\mathcal{G}$ of linear orderings of ω which embed in $\mathcal{S}$ is $\mathbf{\Sigma}^1_1$. On the other hand, $\mathcal{G}$ is the class of countable well-orderings, and hence Π^1_1 complete, contradiction.)
- More generally, Harrington and Shelah (1982) proved that no Borel linear order has a subset of order type ω_1 (even if equality is presented by a nontrivial equivalence relation).
- Harrington, Shelah, and Marker (1988) showed that any Borel partial order without an uncountable Borel antichain is a (disjoint) union of countably many Borel chains.

A Borel partial order without a Borel linear extension

Proposition (Montalbán and Nies)

There is a Borel relation $R \subseteq X \times X$, where $X = \{0,1\}^\omega$, such that R is a partial order without a Borel linear extension.

Proof.

Let P be any Borel preorder with an ω_1-chain, such as $\mathcal{P}(\omega)$ with almost inclusion $\subseteq^*$. We equip each equivalence class for this preorder with the lexicographical order $\leq_{lex}$. Thus we define

$$Rxy \Leftrightarrow (Pxy \wedge \neg Pyx) \vee (Pxy \wedge Pyx \wedge x \leq_{lex} y),$$

where $x, y \subseteq \omega$. Then R is a Borel partial order that has no Borel linear extension by the result of Harrington and Shelah. □

Classes of structures defined by automata

We also consider classes of structures presented by a type of infinitary automaton, such as Büchi, or Rabin automata.

- Büchi structures are Borel, but Rabin structures not necessarily.
- Such classes have a decidable theory.
- Such classes are closed under first-order interpretations (unlike the Borel structures).

Büchi Automata 1

- A Büchi automaton B has the same components as a (nondeterministic) finite automaton:
  - a (finite) alphabet $\mathbb{A}$
  - a set of states with a subset of accepting states, and an initial state,
  - a transition relation, of the format

$$\langle \text{state, input symbol, new state} \rangle.$$

- The inputs are ω-words, such as $01001000100001\ldots$.

Büchi Automata 2

- A run of Büchi automaton B on an ω-word is a sequence of states consistent with the transition relation when B is "reading" that word.
- B accepts an ω-word if some run of B on this word is infinitely often in an accepting state. B recognizes an ω-language if it accepts precisely the words in it.
  For instance, the set of ω-words over $\{0, 1\}$ with infinitely many 1's is recognizable by a Büchi automaton.

Complementation theorem, Büchi 1966

The Büchi recognizable sets are closed under complements.

Thus, these sets are analytic and co-analytic, and hence Borel.

Büchi structures

- A k-tuple of ω-words over $\mathbb{A}$ can be put into the format of a $k \times \omega$ "table". Thus it corresponds to a single ω-word over $\mathbb{A}^k$.
- A k-ary relation on ω-words over $\mathbb{A}$ is called Büchi recognizable if the corresponding set of tables is Büchi recognizable
- A Büchi presentation is a Borel presentation where the standard Polish space is $\mathcal{X} = \mathbb{A}^\omega$, and the domain and the relevant relations are all Büchi recognizable.

Many examples of Borel structures are in fact Büchi structures:

- the reals with addition
- the 2-adic integers with addition

0	1	1	0	0	0	1	1	0	0	···
1	0	1	0	0	1	1	0	1	0	···
1	1	0	1	0	1	0	0	0	1	···

- the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$.

A fact from descriptive set theory

We need a fact from descriptive set theory:
almost identity cannot be Borel reduced to identity.

- Let $\mathcal{X}, \mathcal{Y}$ be Polish spaces. A function $F : \mathcal{X} \to \mathcal{Y}$ is a Borel function if the preimage $F^{-1}(R)$ is Borel for each open (or Borel) set R.
- Equivalently, the graph of F is Borel as a subset of $\mathcal{S} \times \mathcal{T}$.

Let $=^*$ denote almost equality of subsets of ω.

Fact

There is no Borel function F on Cantor space $\mathcal{P}(\omega)$ such that

$$A =^* B \Leftrightarrow F(A) = F(B) \text{ for each } A, B \subseteq \omega.$$

This can be proved for instance using the 0-1 law for the product measure on $\mathcal{P}(\omega)$.

The Basic Lemma

$(\mathcal{P}(\omega), \subseteq)$ is "Borel stable":

Basic Lemma

Each isomorphism between two Borel presentations of $(\mathcal{P}(\omega), \subseteq)$ has a Borel graph.

This is so because the isomorphism is given by its effect on the countably many atoms.
The lemma also works for non-injective presentations.

A Büchi structure without an injective Borel presentation

Theorem (Hjorth, Khoussainov, Montalbán, Nies, 2008)

There is a Büchi presentable structure $\mathbf{A}$ *without an injective Borel presentation.*

Proof. The signature consists of:

- binary predicates $E, \leq, R$,
- a unary predicate U.

Let $\mathcal{B} = (\mathcal{P}(\omega), \subseteq)$ and $\mathcal{B}^* = (\mathcal{P}(\omega)/\mathrm{Fin}, \leq)$.
Let $\mathbf{A}$ be the disjoint sum of $\mathcal{B}$ and $\mathcal{B}^*$ as partial orders.
Let $U^{\mathbf{A}}$ be the left side, and let $R^{\mathbf{A}}$ be the projection $\mathcal{B} \to \mathcal{B}^*$.

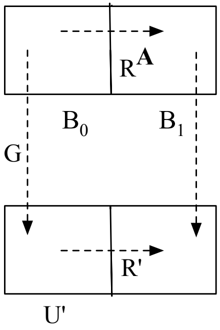
The Büchi presentation is $\mathcal{A} = (\mathcal{B}_0 \sqcup \mathcal{B}_1, E, \leq, \mathcal{B}_0, S)$, where

- E is identity $\mathcal{B}_0$, and almost equality $=^*$ on $\mathcal{B}_1$;
- S is the canonical bijection between the two copies $\mathcal{B}_0, \mathcal{B}_1$ of $\mathcal{B}$.

Now assume that $\mathcal{S} = (D, \leq', U', R')$ is an injective Borel presentation of $\mathbf{A}$. Let $\Phi \colon \mathcal{A} \to \mathcal{S}$ be an isomorphism. Let G be the restriction of Φ to $\mathcal{B}_0$. Then G is Borel by the Basic Lemma. Now, for $X, Y \subseteq \omega$,

$$\begin{aligned} X =^* Y \quad &\Leftrightarrow \quad R^{\mathbf{A}}(X) = R^{\mathbf{A}}(Y) \\ &\Leftrightarrow \quad \Phi(R^{\mathbf{A}}(X)) = \Phi(R^{\mathbf{A}}(Y)) \\ &\Leftrightarrow \quad R'(G(X)) = R'(G(Y)), \end{aligned}$$

contrary to the fact from descriptive set theory.



All F_σ-ideals of $\mathcal{P}(\omega)$ behave similarly

Finkel and Todorcevic gave two Borel Boolean algebras which are isomorphic under CH, but not under the open coloring axiom OCA.

For isomorphism under CH they used this:

Theorem (Just and Krawzcyk, 1984)

Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be a proper F_σ ideal containing the ideal Fin *of finite sets. Then $\mathcal{P}(\omega)/\mathcal{I}$ is a dense $\aleph_1$-saturated Boolean algebra.*

If CH holds then $\aleph_1 = 2^{\aleph_0} =$ the cardinality of $\mathcal{P}(\omega)/\mathcal{I}$.
So this implies that $\mathcal{P}(\omega)/\mathcal{I} \cong \mathcal{P}(\omega)/\text{Fin}$ is the unique saturated dense Boolean algebra of this size.

An F_σ-ideal of $\mathcal{P}(\omega)$

Let $\mathcal{K}$ be the ideal of $\mathcal{P}(\omega \times \omega)$ consisting of the sets A such that

$$\exists n \, [A \subseteq \omega \times n]\}.$$

Fact (following Finkel and Todorcevic 2010)

- *$\mathcal{K}$ is an F_σ ideal, and hence Borel.*
- *This ideal can be recognized by a Rabin automaton.*

Hence $\mathcal{P}(2^{<\omega})/\mathcal{K}$ is a Borel structure. It is also representable by Rabin tree automata, a generalization of Büchi automata.

Their orginal example was the ideal of $\mathcal{P}(2^{<\omega})$ consisting of the sets without an infinite antichain under the prefix relation.

The open coloring axiom

The open coloring axiom (OCA) in the version of Todorcevic (1989) says that each undirected graph on $\mathbb{R} \times \mathbb{R}$ with an open set of edges

- either has an uncountable clique,
- or the vertices can be colored with countably many colors so that no adjacent vertices have the same color.

If ZFC is consistent, then ZFC + OCA is consistent (see Jech, 2002).

Theorem (Todorcevic, 1998)

Assume ZFC + OCA. Let $\mathcal{J}$ be an analytic ideal of $\mathcal{P}(U)$, U countable, such that

$$\mathcal{P}(U)/\mathcal{J} \text{ can be embedded into } \mathcal{P}(\omega)/Fin.$$

Then $\mathcal{J}$ is trivial: there is $S \subseteq U$ such that

$$\mathcal{J} = \{B \subseteq U\colon\ B \cap S \text{ finite}\}.$$

Clearly the ideal $\mathcal{K} = \{A\colon\ \exists n\,[A \subseteq \omega \times n]\}$ is not trivial.
Thus under OCA the Borel presentable Boolean algebras $\mathcal{P}(\omega)/\mathcal{K}$ and $\mathcal{P}(\omega)/\mathrm{Fin}$ are not isomorphic.
Hence, ZFC cannot decide whether they are isomorphic.

Note that these are also Rabin presentable Boolean algebras. No example known (to me) for Büchi presentable BAs

For countable languages, there is a Borel completeness theorem.

Theorem (H. Friedman, 1979)

Each consistent theory T in a countable language has a Borel model of size the continuum. In fact, its elementary diagram is Borel.

Friedman expanded a countable model of T to the size of the continuum by adding order indiscernibles, ordered like the real numbers.

This implies the model has continuum many Borel automorphisms, and only realizes countably many types.

For instance, if T is the theory of algebraically closed fields of characteristic 0, then one obtains a Borel presentation of the field $\mathbb{C}$.

This Borel presentation of $\mathbb{C}$ is not Borel isomorphic to the natural one, because the natural one has only two Borel automorphisms (observation of Nies and Shore, 2009).

Borel models realizing many types

The following 1987 notes by Knight and Simpson
give a different kind of continuum size model for countable "tidy" theories.

Model-theoretic proof of Woodin's Theorem

Woodin gave a set-theoretic proof of the following theorem. Woodin also gave a reduction of the full theorem to the special case $T = TA$.

tidy: built in Skolem fu

Theorem: Let T be a countable complete tidy theory, and let $\mathfrak{T}$ be a perfect tree of types. Then T has a Borel model realizing $2^{\aleph_0}$ of the types represented paths through $\mathfrak{T}$.

Sample application

Let p_n be the n-th prime number.

For $S \subseteq \omega$ consider the type

$$\tau_S(x) = \{p_n \mid x \colon n \in S\} \cup \{p_n \nmid x \colon n \notin S\}.$$

There is a Borel model of $\mathrm{Th}(\mathbb{N}, +, \times)$ in which uncountably many of these types are realized.

Borel Signatures

Definition

A Borel signature is a Borel set $\mathcal{L}$ consisting of relation and function symbols, such that the sets

$$\{R \in \mathcal{L} : R \text{ is a relation symbol of arity } n\}$$

and

$$\{f \in \mathcal{L} : f \text{ is a function symbol of arity } n\}$$

are all Borel.

The corresponding first-order language can be viewed as a Borel set in a suitable standard Polish space. So we have a notion of Borel theories.

Borel presentations over Borel signatures

Suppose we are given an (uncountable) Borel signature $\mathcal{L}$.
For a Borel presentation of a structure, we now need to require that the relations and functions are uniformly Borel.

Main Definition, extended to Borel signatures

An $\mathcal{L}$-structure $\mathbf{A}$ is called Borel if there is a function ϕ from a Borel set D in standard Borel space $\mathcal{X}$ onto $\mathbf{A}$, such that for each n,

$$\{(R, d_1, \ldots, d_n)\colon\ R^{\mathbf{A}}\phi(d_1), \ldots, \phi(d_n)\} \subseteq \mathcal{L} \times D^n \text{ is Borel,}$$

where R ranges over n-ary relation symbols in $\mathcal{L}$;
a similar condition holds for the n-ary function symbols.

A consistent Borel theory without a Borel model or Borel completion 1

Example

The following is a Borel signature:
for each $X \subseteq \omega$ a constant symbol c_X ; a unary predicate U.

Theorem (Hjorth and Nies, JSL, June 2011)

In the signature given above, there exists a consistent Borel theory with no Borel completion.

A consistent Borel theory without a Borel model or Borel completion 2

The Borel theory expresses that $\{X : U(c_X)\}$ is a filter on ω containing all the co-finite sets.
In more detail, it consists of the sentences

- $c_A \neq c_B$, for every $A \neq B \subseteq \omega$;
- $U(c_A) \to U(c_B)$, for every pair of sets such that $A \subseteq B \subseteq \omega$;
- $U(c_A) \leftrightarrow \neg U(c_{\omega \setminus A})$, for every $A \subseteq \omega$;
- $U(c_A) \wedge U(c_B) \to U(c_{A \cap B})$, for every $A, B \subseteq \omega$;
- $\neg U(c_A)$, for each finite set $A \subseteq \omega$.

Any completion $\mathcal{S}$ of the theory determines the free ultrafilter $\{A \colon \mathcal{S} \vdash U(c_A)\}$ of ω, and hence cannot be Borel.
A similar argument works for Borel models.

The completeness theorem fails in the general Borel setting

Theorem (Hjorth and Nies, JSL, June 2011)

There exists a consistent complete Borel theory T with no Borel model.

At its core this uses:

Fact from descriptive set theory

There is no Borel reduction of almost equivalence $=^*$ to the equality relation on a Cantor space 2^ω.

T is defined in such a way that any Borel model of T would provide such a reduction.

Borel models of T would be ω models

T has several sorts. One of them is $\mathcal{N}$, standing for natural numbers. On $\mathcal{N}$ there are

constants e_n $(n \in \omega)$ and

unary predicate symbols U_x $(x \in 2^\omega)$.

T expresses that the e_n are pairwise distinct, and for each $e \in \mathcal{N}$, the truth value of $U_x(e)$ behaves the way one expects for the set x. For instance, whenever $z \subseteq y$ we have a sentence $\forall e \in \mathcal{N}\,(U_z(e) \Rightarrow U_y(e))$ in T. For $z = \{n\}$ we have a sentence $\forall e \in \mathcal{N}\,(U_z(e) \Leftrightarrow e = e_n)$.

Then any Borel model $\mathbf{A}$ of T would be an ω model, i.e., $\mathcal{N}^{\mathbf{A}} = \{e_n^{\mathbf{A}} \colon n \in \omega\}$. Else there would be a free Borel ultrafilter on ω.

- Once we have ensured that $\mathcal{N}$ corresponds to the natural numbers, we can also have a sort for the tree $2^{<\omega}$, and one for paths through it. So we have a sort $\mathcal{B}$ denoting Cantor space 2^{ω}.
- There are further sorts $\mathcal{C}, \mathcal{F}$. There is an equivalence relation E on $\mathcal{C}$ thought of as almost equivalence $=^*$. The sort $\mathcal{F}$ denotes 1-1 functions $\mathcal{C}/E \to \mathcal{B}$.
- T is complete and Borel because all countable models of its restrictions to countable sub-signatures (sharing some fixed finite collection of base symbols) are isomorphic.
- This uses Malitz' Lemma that all countable dense sets of paths in the tree $2^{<\omega}$ are tree automorphic.
- Part of the difficulty is that we also have to rule out non-injective Borel presentations.

Open Questions

- The Borel dimension of a Borel structure is the number of equivalence classes modulo Borel isomorphism on the set of its Borel presentations. Is there a structure of Borel dimension strictly between 1 and the continuum? (HKMN 2008)

  For instance, $(\mathbb{R}, +)$ has Borel dimension 2^ω, while $(\mathbb{R}, +, \times)$ has Borel dimension 1. Does the field $\mathbb{C}$ have Borel dimension 2^ω? (Nies and Shore, recent)

- If a Scott set is Borel, is it already the standard system of a Borel model of Peano arithmetic? (Woodin, 1990s)

  Background: Scott showed that the countable Scott sets are precisely the standard systems of countable models of Peano arithmetic. Knight and Nadel (1982) proved the analogous result for Scott sets and models of the size ω_1. For the general uncountable case, the analogous statement is open.

- Does the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ have an injective Borel presentation? (HKMN 2008)
  Analogous question for $Sym(\omega)/\mathrm{Fin}$, Calkin Algebra.

- Does every Borel field F have an algebraically closed Borel extension? (Nies, 2009; variant appeared in D. Markers March 2010 FOM posting.)

  If not, this would yield a further, more natural, example of a complete Borel theory without a Borel model:

$$ACF_m \cup \mathrm{Diag}_F,$$

  where m is the characteristic of F, and Diag_F its atomic diagram. One thing one could try would be to code the Hjorth/Nies Borel theory into a Borel field.

References (all available on my web page)

From automatic structures to Borel structures by Hjorth, Khoussainov, Montalban, and Nies, LICS 2008.

Borel structures and Borel theories by Hjorth and Nies, JSL, June 2011.

Borel structures: a brief survey by Montalbán and Nies, to appear in "Effective Mathematics of the Uncountable", ASL Lecture Notes in Logic, 2012.

These and similar slides, on my web page.