Borel structures

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Presentations of continuum size structures

Continuum size structures occur naturally in analysis, algebra, and other areas. Examples are the additive groups of real numbers, or of p-adic integers, and the ring of continuous functions on the unit interval.

What can we say about their complexity?

- We will study continuum size structures that are effective in the sense that they have a Borel presentation.
- This includes the examples given above.
- It actually includes most continuum size structures from mathematics.

The plan

- We define Borel presentations of structures. We give examples.
- We consider the interplay of Borel theories and Borel structures, and show that the completeness theorem from logic fails in the Borel setting for an uncountable signature.
- We end with open questions.

Borel sets

- Let $\mathcal{X}$ be a "standard Borel space", for instance $\mathbb{R}$, Cantor space $\{0, 1\}^\omega$, or, more generally, some uncountable Polish topological space (that is, complete metrizable, and separable).
- The Borel sets of $\mathcal{X}$ are the smallest σ-algebra containing the open sets.
- Thus, the Borel sets are the sets obtained from the open sets by iterated applications of complementation and countable union.

Sets that are Borel, and sets that are not Borel

- Most sets of reals one considers in analysis are Borel.
- The set of wellorderings on ω is not Borel.
- A free ultrafilter on ω (viewed as a subset of Cantor space 2^ω) is not Borel (not even measurable).

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Larger classes of sets

$\boldsymbol{\Sigma}^1_1$ (or analytic)	projections of Borel relations
$\boldsymbol{\Pi}^1_1$ (or co-analytic)	complements of $\boldsymbol{\Sigma}^1_1$ sets

Souslin (1917) proved that

$$\text{Borel} = \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1.$$

Computable structures

Computable model theory studies the effective content of model theoretic concepts and results for countable structures.

Definition

A structure $\mathbf{A}$ in a finite signature is called computable if

- there is a function from a computable set $D \subseteq \omega$ onto $\mathbf{A}$, such that
- the preimages of the atomic relations of $\mathbf{A}$ (including equality) are computable.

Example

$(\mathbb{Q}, <)$ and $(\omega, +, \times)$ are computable.

Main Definition (H. Friedman, 1979)

A structure $\mathbf{A}$ in a finite signature is called Borel structure if

- there is a function from a Borel set D in a standard Borel space $\mathcal{X}$ onto $\mathbf{A}$, such that
- the preimages of the atomic relations of $\mathbf{A}$ (including equality) are Borel relations on $\mathcal{X}$.

We call the tuple consisting of D and these preimages of the atomic relations a Borel presentation of $\mathbf{A}$.

- We allow that an element of $\mathbf{A}$ is represented by a whole equivalence class of a Borel equivalence relation E.
  If E is the identity relation on D, we say the Borel presentation is injective.
- Injectivity is free for computable structures, because in that case the equivalence relation has a computable separating set.

Examples of Borel structures in algebra

Many algebraic structures of size the continuum are Borel.

- The Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ is Borel, where Fin denotes the ideal of finite sets.
- For a countable structure S in a countable signature, the following structures are Borel:
  - the lattice of substructures of S,
  - the congruence lattice of S,
  - the automorphism group of S.

These structures are in fact arithmetical in the atomic diagram of S.

Borel structures in analysis

Most structures from analysis of size the continuum are Borel.
For instance, the fields $(\mathbb{R}, +, \times)$, $(\mathbb{C}, +, \times)$, the ring $\mathcal{C}[0,1]$ are Borel.

Fact (Hjorth and Nies, 2010)

There are continuum many non-Borel isomorphic injective Borel presentations of $(\mathbb{R}, +)$.

Proof. For each $p > 1$, we obtain a Borel presentation of $(\mathbb{R}, +)$ as the abelian group underlying the Banach space

$$\ell^p = \{\vec{x} \in \mathbb{R}^\omega : \sum_n |x_n|^p < \infty\},$$

where the norm is $\|\vec{x}\|_p = (\sum_n |x_n|^p)^{1/p}$.
Two different Borel presentations of this sort are not Borel isomorphic, because any two Borel isomorphic Polish groups are already homeomorphic. (See Thm. 9.10 in Kechris, Classical descriptive set theory.)

Restrictions on Borel orders

- The well-ordering of length $2^{\aleph_0}$ is **not** Borel.
  (If $\mathcal{S}$ is a Borel presentation of this well-ordering, then the class $\mathcal{G}$ of linear orderings of ω which embed in $\mathcal{S}$ is $\mathbf{\Sigma}^1_1$. On the other hand, $\mathcal{G}$ is the class of countable well-orderings, and hence Π^1_1 complete, contradiction.)
- More generally, Harrington and Shelah (1982) proved that no Borel linear order has a subset of order type ω_1 (even if equality is presented by a nontrivial equivalence relation).
- Harrington, Shelah, and Marker (1988) showed that any Borel partial order without an uncountable Borel antichain is a (disjoint) union of countably many Borel chains.

Classes of structures defined by automata

To differentiate the complexity of structures, we consider classes of structures presented by a type of infinitary automaton, such as Büchi, or Rabin automata.

- Büchi structures are Borel, but Rabin structures not necessarily.
- The classes have a decidable theory.
- The classes are closed under first-order interpretations (unlike the Borel structures).

Büchi Automata 1

- A Büchi automaton *B* has the same components as a (nondeterministic) finite automaton:
  - a (finite) alphabet $\mathbb{A}$
  - a set of states with a subset of accepting states, and an initial state,
  - a transition relation, of the format

$$\langle \text{state, input symbol, new state} \rangle.$$

- The inputs are ω-words, such as 010010001000001 . . ..

Büchi Automata 2

- A run of Büchi automaton B on an ω-word is a sequence of states consistent with the transition relation when B is "reading" that word.
- B accepts an ω-word if some run of B on this word is infinitely often in an accepting state.
  For instance, the set of ω-words over $\{0, 1\}$ with infinitely many 1's is recognizable by a Büchi automaton.

Complementation theorem, Büchi 1966

The Büchi recognizable sets are closed under complements.

Thus, these sets are analytic and co-analytic, and hence Borel.

Büchi structures

- A k-tuple of ω-words over $\mathbb{A}$ can be put into the format of a $k \times \omega$ "table". Thus it corresponds to a single ω-word over $\mathbb{A}^k$.
- A k-ary relation on ω-words over $\mathbb{A}$ is called Büchi recognizable if the corresponding set of tables is Büchi recognizable
- A Büchi presentation is a Borel presentation where the standard Polish space is $\mathcal{X} = \mathbb{A}^\omega$, and the domain and the relevant relations are all Büchi recognizable.

Many examples of Borel structures are in fact Büchi structures:

- the reals with addition
- the 2-adic integers with addition

0	1	1	0	0	0	1	1	0	0	⋯
1	0	1	0	0	1	1	0	1	0	⋯
1	1	0	1	0	1	0	0	0	1	⋯

- the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$.

Muller automata

A Muller automaton is the same as a Büchi automaton, except that acceptance is given by a nonempty set $\mathcal{F}$ of sets of states.

A run of a Muller automaton is accepting if

the set of states that occurs infinitely often is in $\mathcal{F}$.

Büchi automata recognize the same ω-languages as deterministic Muller automata M.

The property

"the state s occurs infinitely often in the run of M on w"

is Π^0_2. So every Büchi recognizable set is not only Borel, but in fact a Boolean combination of Π^0_2 (and hence G_δ) sets.

Rabin-Muller automata

- A Rabin-Muller automaton M over the alphabet $\mathbb{A}$ is like a Muller automaton, but instead of ω-words, it processes full infinite binary trees, where the nodes are labeled with symbols from $\mathbb{A}$.
- The transition relation is now of the format

$$\langle \text{ state, input symbol, new left state, new right state } \rangle.$$

- M accepts a labeled tree x if there is a run of M on x such that along each path, the set of states occurring infinitely often is in $\mathcal{F}$.
- By definition, each set recognizable by a Rabin-Muller automaton is Σ^1_2:

  "there is a run such that for each path [... (arithmetical)].

- Hence the set is Δ^1_2 by Rabin's complementation theorem.
- The domain of Rabin presentations consists of labeled trees. We define Rabin structures in the same way as we defined Büchi structures.

The category algebra is Rabin presentable

An open set $U \subseteq \{0,1\}^\omega$ is called regular open if $U = \overset{\circ}{\overline{U}}$.

- The regular open sets form a Boolean algebra *RO*.
- *RO* is the completion of the countable dense Boolean algebra.
- *RO* is isomorphic to the category algebra of an uncountable Polish space (i.e. the sets with Baire property, modulo the meager sets).

Theorem

RO is injective Rabin presentable.

Proof. We represent open sets in 2^ω by sets of strings closed under extension. A Rabin automaton can recognize their inclusion.

The regular open sets are first-order definable (and hence Rabin recognizable) because U is regular open $\Leftrightarrow$

U contains all the open sets W that are disjoint with the same open sets as U.

Two F_σ-ideals

Theorem (Just and Krawzcyk, 1984)

Let $\mathcal{I} \subseteq \mathcal{P}(\omega)$ be a (proper) F_σ ideal containing Fin.
Then $\mathcal{P}(\omega)/\mathcal{I}$ is a dense $\aleph_1$-saturated Boolean algebra.

Under CH, this implies that $\mathcal{P}(\omega)/\mathcal{I} \cong \mathcal{P}(\omega)/\mathrm{Fin}$.

Let $\mathcal{K}$ be the ideal of $\mathcal{P}(2^{<\omega})$ consisting of the sets of strings that have no infinite antichain.

Theorem (Finkel and Todorcevic, 2010)

- *$\mathcal{K}$ is an F_σ ideal.*
- *This ideal can be recognized by a Rabin-Muller automaton.*

Hence $\mathcal{P}(2^{<\omega})/\mathcal{K}$ is both a Borel structure, and a Rabin structure.

The open coloring axiom

The open coloring axiom (OCA, Todorcevic, 1989) says that each undirected graph on $\mathbb{R} \times \mathbb{R}$ with an open set of edges

- either has an uncountable clique,
- or can be colored with countably many colors.

If ZFC is consistent, then both ZFC + CH, and ZFC + OCA are consistent (see Jech, 2002).

Theorem (Finkel and Todorcevic, 2010)

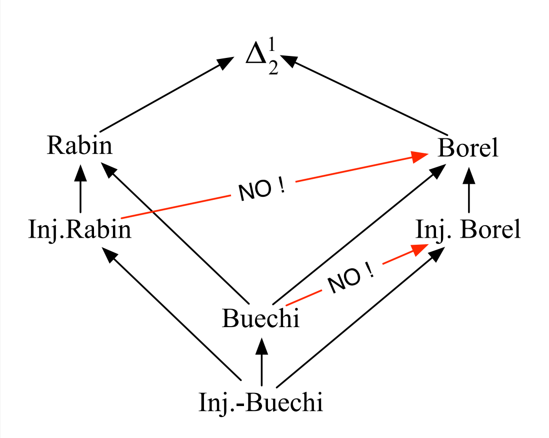
Under ZFC + OCA

$$\mathcal{P}(2^{<\omega})/\mathcal{K} \text{ is not isomorphic to } \mathcal{P}(2^{<\omega})/\mathrm{Fin}.$$

(In fact, it is not even embeddable into $\mathcal{P}(2^{<\omega})/\mathrm{Fin}$.)

Conclusion: ZFC cannot decide isomorphism of Borel Boolean algebras, or Rabin Boolean algebras.

Separating classes of structures (Hjorth, Khoussainov, Montalbán, Nies, 2008)



Complexity classes of continuum sized structures

The Basic Lemma

$(\mathcal{P}(\omega), \subseteq)$ is “Borel stable”:

Basic Lemma

Each isomorphism between two Borel presentations of $(\mathcal{P}(\omega), \subseteq)$ has a Borel graph.

This is so because the isomorphism is given by its effect on the countably many atoms. It also works for non-injective presentations.

A fact from descriptive set theory

To see that some Borel structure $\mathbf{A}$ has no injective Borel representation, we need a fact from descriptive set theory: almost identity cannot be Borel reduced to identity.

- Let $\mathcal{X}, \mathcal{Y}$ be Polish spaces. A function $F : \mathcal{X} \to \mathcal{Y}$ is a Borel function if the preimage $F^{-1}(R)$ is Borel for each open (or Borel) set R.
- Equivalently, the graph of F is Borel as a subset of $\mathcal{S} \times \mathcal{T}$.

Let $=^*$ denote almost equality of subsets of ω.

Fact

There is no Borel function F on Cantor space $\mathcal{P}(\omega)$ such that

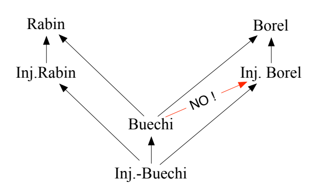
$$A =^* B \Leftrightarrow F(A) = F(B) \text{ for each } A, B \subseteq \omega.$$

This uses that each Borel function is continuous on a comeager set.

Büchi, not injective Borel

Theorem (HKMN, 2008)

There is a Büchi presentable structure **A** *without an injective Borel presentation.*



Proof. The signature consists of:

- binary predicates $E, \leq, R$,
- a unary predicate U.

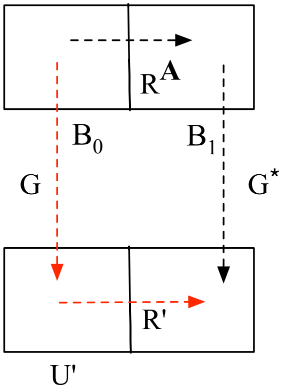
Let $\mathcal{B} = \mathcal{P}(\omega)$ and $\mathcal{B}^* = \mathcal{P}(\omega)/\mathrm{Fin}$, viewed as partial orders. Let **A** be the disjoint sum of $\mathcal{B}$ and $\mathcal{B}^*$.
Let $U^{\mathbf{A}}$ be the left side, and let $R^{\mathbf{A}}$ be the projection $\mathcal{B} \rightarrow \mathcal{B}^*$.

The Büchi presentation is $\mathcal{A} = (\mathcal{B}_0 \sqcup \mathcal{B}_1, E, \leq, \mathcal{B}_0, S)$, where

- E is identity $\mathcal{B}_0$, and almost equality $=^*$ on $\mathcal{B}_1$;
- S is the canonical bijection between the two copies $\mathcal{B}_0, \mathcal{B}_1$ of $\mathcal{B}$.

Summarize: the Büchi presentation of **A** is $\mathcal{A} = (\mathcal{B}_0 \sqcup \mathcal{B}_1, E, \leq, \mathcal{B}_0, S)$.
Now assume for contradiction that $\mathcal{S} = (D, =, \leq', U', R')$ is an injective Borel presentation of **A**.
Let G be the restriction to $\mathcal{B}_0$ of the isomorphism $\mathcal{A} \to \mathcal{S}$. Then G is a Borel function by the Basic Lemma.
Let G^* be the restriction to $\mathcal{B}_1$ of this isomorphism.

The diagram commutes.
Hence $R' \circ G$ is a Borel reduction of $=^*$ to the identity, contradiction.



A structure that is not Borel

Theorem (Hjorth, Khoussainov, Montalbán and N, 2008)

Suppose C is a countable set and $U \subseteq \mathcal{P}(C)$ is $\mathbf{\Pi}^1_1$ but not Borel. Then the structure $(\mathcal{P}(C), \subseteq, U)$ has no Borel presentation.

Proof. Suppose $(A, E, \leq, V)$ is a Borel presentation such that

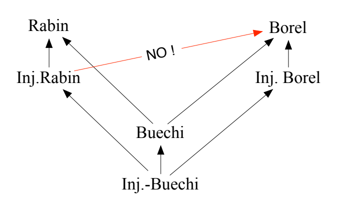
$$\Psi\colon (\mathcal{P}(C), \subseteq, U) \cong (A, E, \leq, V)/E.$$

Then $X \in U \Leftrightarrow [\Psi(X)]_E \in V/E \Leftrightarrow \exists b \in V \left[\Psi(X) = [b]_E\right]$.

The map Ψ has a Borel graph by the basic lemma.
So U is $\mathbf{\Sigma}^1_1$, hence Borel. Contradiction.

Theorem (HKMN, 2008)

Some injective Rabin structure is not Borel.



Proof. Let $C = 2^{<\omega}$.

- By Niwinsky (1985), the language

  $$U = \{T \subseteq 2^{<\omega} : \forall Z \in 2^{\omega} \, [T \text{ has only finitely many 1s along } Z]\}$$

  is Rabin recognizable, Π^1_1, but not Borel.

  - To see this, consider the embedding of $\omega^{<\omega}$ into $2^{<\omega}$ given by

    $$n_0, \ldots, n_k \to 0^{n_0}1 \ldots 0^{n_k}1.$$

  - The pre-image of U under this embedding is the class of well-founded trees, which is Π^1_1, but not Borel.

- Hence $(\mathcal{P}(C), \subseteq, U)$ is injective Rabin, but not Borel.

For countable languages, there is Borel completeness theorem.

Theorem (H. Friedman, 1979)

Each consistent theory T in a countable language has a Borel model of size the continuum. In fact, its elementary diagram is Borel.

Friedman expanded a countable model of T to the size of the continuum by adding order indiscernibles, ordered like the reals numbers.

This implies the model has continuum many Borel automorphisms.

For instance, if T is the theory of algebraically closed fields of characteristic 0, then one obtains a Borel presentation of the field $\mathbb{C}$.

This yields a Borel presentation of the field $\mathbb{C}$ not Borel isomorphic to the natural one, because the natural one has only two Borel automorphisms (observation of Nies and Shore, 2009).

Borel Signatures

Definition

A Borel signature is a Borel set $\mathcal{L}$ consisting of relation and function symbols, such that the sets

$$\{R \in \mathcal{L} : R \text{ is a relation symbol of arity } n\}$$

and

$$\{f \in \mathcal{L} : f \text{ is a function symbol of arity } n\}$$

are all Borel.

The corresponding first-order language can be viewed as a Borel set in a suitable standard Polish space. So we have a notion of Borel theories.

A consistent Borel theory without a Borel completion

Example

The following is a Borel signature:
for each $X \subseteq \omega$ a constant symbol c_X ; a unary predicate U.

Theorem (Hjorth and Nies, JSL, to appear)

In the signature, given above, there exists a consistent Borel theory with no Borel completion.

The Borel theory expresses that

$$\{X \colon U(c_X)\}$$

is a filter on ω containing all the co-finite sets. A Borel completion of the theory would determine a Borel free ultrafilter on ω, contradiction.

Borel presentations over Borel signatures

Suppose we are given an (uncountable) Borel signature $\mathcal{L}$.
For a Borel presentation of a structure, we now require that the relations and functions are uniformly Borel:

Definition

An $\mathcal{L}$-structure $\mathbf{A}$ is called Borel if there is a function ϕ from a Borel set D in standard Borel space $\mathcal{X}$ onto $\mathbf{A}$, such that for each n,

$$\{(R, d_1, \ldots, d_n) \colon\ R^{\mathbf{A}}\phi(d_1), \ldots, \phi(d_n)\} \subseteq \mathcal{L} \times D^n \text{ is Borel,}$$

where R ranges over n-ary relation symbols in $\mathcal{L}$;
a similar condition holds for the n-ary function symbols.

The completeness theorem fails in this general Borel setting.

Theorem (Hjorth and Nies, JSL, to appear)

There exists a consistent complete Borel theory T with no Borel model.

- At its core this uses the fact from descriptive set theory that there is no Borel reduction of $=^*$ to the equality relation on a standard space. T is defined in such a way that any Borel model of T would provide such a reduction.
- T is complete and Borel because isomorphic are: countable models of its restrictions to countable sub-signatures (sharing some fixed collection of base symbols).
- This uses Malitz' Lemma that all countable dense sets of paths in the tree $2^{<\omega}$ are tree automorphic.
- Part of the difficulty is that we also have to rule out non-injective Borel presentations.

Open Questions

- The Borel dimension of a Borel structure is the number of equivalence classes modulo Borel isomorphism on the set of its Borel presentations. Is there a structure of Borel dimension strictly between 1 and the continuum? (HKMN 2008)

  For instance, $(\mathbb{R}, +)$ has Borel dimension 2^ω, while $(\mathbb{R}, +, \times)$ has Borel dimension 1. Does the field $\mathbb{C}$ have Borel dimension 2^ω? (Nies and Shore, recent)

- If a Scott set is Borel, is it already the standard system of a Borel model of Peano arithmetic? (Woodin, 1990s)

  Background: Scott showed that the countable Scott sets are precisely the standard systems of countable models of Peano arithmetic. Knight and Nadel (1982) proved the analogous result for Scott sets and models of the size ω_1. For the general uncountable case, the analogous statement is open.

- Does the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ have an injective Borel presentation? (HKMN 2008)
- Characterize the Büchi Boolean algebras and the Rabin Boolean algebras.
- Does every Borel field F have an algebraically closed Borel extension?

  If not, this would yield a further, more natural, example of a complete Borel theory without a Borel model:

$$ACF_m \cup \mathrm{Diag}_F,$$

  where m is the characteristic of F, and Diag_F its atomic diagram. (Marker, 2010; Nies, 2009)

References (all available on my web page)

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From automatic structures to Borel structures by Hjorth, Khoussainov, Montalban, and Nies, LICS 2008.

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Borel structures: a brief survey by Montalbán and Nies, to appear in EMU proceedings.

These slides, on my web page.