K-triviality in computable metric spaces

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K-trivial sets of natural numbers

- The property of K-triviality for a set of natural numbers was introduced by Chaitin and Solovay 1975. It was intensely studied during the last decade. Today it is a key notion at the interface of computability and randomness.
- Surprising coincidence results have been obtained. K-trivial sets are at the same time
  - far from random (by definition)
  - weak as an oracle set
  - computably approximable with a small total of changes.
- We will extend the notion of K-triviality to the more general setting of points in a computable metric space $\mathcal{M}$.

Main results

- Existence and preservation:
  - Every perfect computable metric space $\mathcal{M}$ contains a K-trivial non-computable point.
  - K-triviality is preserved under computable maps between metric spaces.

- The definition of K-triviality of a point $x \in \mathcal{M}$ is via Cauchy names, generalizing the definition of computable points.

  It is equivalent to an apparently weaker "local" condition stating that special points (something like rationals) close to x are highly compressible.

1. Basics

2. Existence and preservation results for K-trivials

3. A local condition characterizing K-trivials

Prefix-free machines

A partial computable function from binary strings to binary strings is called prefix-free machine if its domain is an anti-chain under the prefix relation of strings.

There is a universal prefix-free machine $\mathbb{U}$: for every prefix-free machine M,

$$M(\sigma) = y \text{ implies } \mathbb{U}(\tau) = y,$$

for a string τ that is only by a constant d_M longer than σ.

Descriptive string complexity K

- The prefix-free Kolmogorov complexity is the length of a shortest $\mathbb{U}$-description of y:

$$K(y) = \min\{|\sigma| \colon \mathbb{U}(\sigma) = y\}.$$

- One can show that $2^{-K(y)}$ is proportional to

$$\lambda\{X \in 2^{\mathbb{N}} \colon \mathbb{U}(\sigma) = y \text{ for some initial segment } \sigma \text{ of } X\},$$

  where λ denotes product measure in Cantor space $2^{\mathbb{N}}$. Informally, this is the probability that $\mathbb{U}$ prints y. This only works with prefix-free machines.

Definition of K-triviality

For a string y, up to constants, $K(|y|) \leq K(y)$, since we can compute $|y|$ from y (here we write numbers in binary). Converse?

Definition

A set A is K-trivial if there is a constant $b \in \mathbb{N}$ such that for each n,

$$K(A \restriction_n) \leq K(n) + b,$$

namely, the K complexity of all initial segments is minimal.

This is opposite to ML-randomness:

- Z is ML-random if all complexities $K(Z \restriction_n)$ are near the upper bound $n + K(n)$, while
- Z is K-trivial if they have the minimal possible value $K(n)$ (all within constants).

Why prefix free machines?

- C-triviality is the analogous notion defined with the usual Kolmogorov complexity C instead of K.
- Each C-trivial set is computable (Chaitin, 1975).
- (Chaitin, 1976) proved that the K-trivial sets are Δ^0_2 (i.e., computed by the halting problem).
- Solovay (1975) was the first to construct a non-computable K-trivial A, which was Δ^0_2 as expected but not computably enumerable.
- Later on, various constructions of a computably enumerable example appeared. E.g., the cost function construction of Downey et al. (2002).

K-trivials are Turing below the halting problem

Theorem

There is a constant $\mathbf{c} \in \mathbb{N}$ such that for each constant $b \in \mathbb{N}$, for each length n

$$|\{x\colon\ |x| = n \wedge K(x) \leq K(n) + b\}| < 2^{\mathbf{c}} 2^{b}.$$

Using this, one proves:

Theorem

(i) For each b, at most $2^{\mathbf{c}+b}$ sets are K-trivial with constant b.
(ii) Each K-trivial set is in Δ^0_2.

Closure under $\oplus$

Theorem (Downey et al., 2002)

If sets A and B are K-trivial then $A \oplus B = 2A \cup 2B + 1$ *is K-trivial.*

More specifically, if both A and B are K-trivial via b, then $A \oplus B$ *is K-trivial via* $3b + \mathcal{O}(1)$.

K-trivial functions $f\colon \mathbb{N} \to \mathbb{N}$

For a number $v \in \mathbb{N}$ (seen as a binary string), recall that $K(v)$ denotes the prefix-free Kolmogorov complexity of v: $K(v) = \min\{|\sigma|\colon \mathbb{U}(\sigma) = v\}$, where $\mathbb{U}$ is a universal prefix-free machine.

Definition

A function $f\colon \mathbb{N} \to \mathbb{N}$ is called K-trivial if there is a constant $b \in \mathbb{N}$ such that for each n, $K(f\restriction_n) \leq K(n) + b$.
Here $f\restriction_n$ denotes the tuple of the first n values of f. We assume some effective encoding of tuples over $\mathbb{N}$ by natural numbers.

This extends the usual definition for sets (seen as $0,1$-valued functions). Each computable function is K-trivial, but not conversely.

Proposition (uses that K-trivial = low for K)

A function f is K-trivial $\iff$ the graph of f is K-trivial.

Computable metric spaces

Definition

Let (M, d) be a complete metric space, and let $(\alpha_i)_{i \in \mathbb{N}}$ be a dense sequence in M.

- $\mathcal{M} = (M, d, (\alpha_i)_{i \in \mathbb{N}})$ is a computable metric space if $d(\alpha_i, \alpha_k)$ is a computable real uniformly in i, k.
- We call the elements of the sequence $(\alpha_i)_{i \in \mathbb{N}}$ the special points. We often identify α_i with $i \in \mathbb{N}$.

Definition

- A sequence $(p_s)_{s \in \mathbb{N}}$ of special points is called a Cauchy name if $d(p_s, p_t) \leq 2^{-s}$ for each $s, t \in \mathbb{N}$, $t \geq s$.
- Since $\mathcal{M}$ is complete, $x = \lim_s p_s$ exists. We say that $(p_s)_{s \in \mathbb{N}}$ is a Cauchy name for x. Note that $d(x, p_s) \leq 2^{-s}$.

K-trivial points

Let $\mathcal{M}$ be a computable metric space. Recall that a point $x \in M$ is called computable if it has a computable Cauchy name.

Definition

A point $x \in M$ is called K-trivial if it has a K-trivial Cauchy name.

- The unit interval, and Baire space $\mathbb{N}^{\mathbb{N}}$ with the ultrametric distance function $d(f, g) = \max\{2^{-n}\colon\ f(n) \neq g(n)\}$, form computable metric spaces in a natural way.
- It is easy to check that in these spaces, a point is K-trivial iff it is K-trivial in the usual sense.

Preservation of K-triviality

A map F between computable metric spaces is called computable if there is an oracle computation procedure that with a Cauchy name for x as an oracle computes a Cauchy name for $F(x)$.

Proposition

Let $\mathcal{M}, \mathcal{N}$ be computable metric spaces, and let the map $F\colon \mathcal{M} \to \mathcal{N}$ be computable. If x is K-trivial in $\mathcal{M}$, then $F(x)$ is K-trivial in $\mathcal{N}$.

This relies on a hard result, the downward closure under $\leq_T$ of the class of K-trivial functions. However, the result can be verified directly if F is Lipschitz. It shows that K-triviality is invariant under the change of computable structure to an equivalent one.

Another preservation fact: if $\mathcal{M}$ is a computable Banach space, then the K-trivial points form a subspace.

A metric space where all K-trivial points are computable

Example

There is a computable metric space $\mathcal{M}$ with a noncomputable point such that the only K-trivial points are the computable points.

Proof.

Let

$$M = \{\Omega_s \colon s \in \mathbb{N}\} \cup \{\Omega\}$$

with the metric inherited from the unit interval,
and with the computable structure given by $\alpha_s = \Omega_s$.
If g is a Cauchy name for Ω then $\Omega \leq_{\mathrm{T}} g$, so g is not K-trivial. □

Existence of K-trivials

In the following fix a computable metric space $\mathcal{M}$ with no isolated points.

Theorem (Brattka and Gherardi, 2009)

There is a computable injective map $F\colon \{0,1\}^{\mathbb{N}} \to \mathcal{M}$ which is Lipschitz.

Corollary

$\mathcal{M}$ contains a K-trivial, non-computable point.

Proof of Corollary.

Let $A \in \{0,1\}^{\mathbb{N}}$ be a K-trivial non-computable set. Then $F(A)$ is K-trivial. The inverse of F is computable on its domain. Hence the point $F(A)$ is non-computable. □

A K-trivial in ℓ_2

It is interesting to look at separable Banach spaces, because they often have a natural computable structure (see book by Pour-El / Richards, 1989).

Examples of such spaces are $\mathcal{C}[0,1]$ with the sup norm, and the sequence spaces ℓ_p for $1 \leq p < \infty$ a computable real.

We give an example of a K-trivial point in Hilbert space ℓ_2 which is not obtained through a Brattka/Gherardi embedding.

Let $\mathbf{e}_0, \mathbf{e}_1, \dots$ be the usual orthonormal basis of ℓ_2.

Example

- Let $g \colon \mathbb{N} \to \mathbb{N}$ be an increasing, non-computable K-trivial function.
- Let $\mathbf{x} = \sum_i 2^{-g(i)} \mathbf{e}_i$. Then $\mathbf{x}$ is non-computable.
- Let $f(n) = \sum_{i=0}^{2n+1} 2^{-g(i)} \mathbf{e}_i$. Then f is a K-trivial Cauchy name for $\mathbf{x}$.

K-trivial compact sets

- Given a Polish space M, let $\mathbb{K}(M)$ denote the Polish space of compact subsets of M with the Hausdorff distance (the maximum distance that a point in one set can have from the other set).
- If M is a computable metric space then $\mathbb{K}(M)$ carries a natural computable structure where the special points are the finite sets of special points in M. Thus we have a notion of K-trivial compact sets.

If M is Cantor space $\{0,1\}^{\mathbb{N}}$, as a computable structure we can take equivalently the clopen sets.

Barmpalias, Cenzer, Remmel and Weber (2009) studied a notion of K-triviality in $\mathbb{K}(\{0,1\}^{\mathbb{N}})$. They call a closed set $\mathcal{C} \subseteq \{0,1\}^{\mathbb{N}}$ is K-trivial if the corresponding tree consisting of strings σ with $[\sigma] \cap \mathcal{C} \neq \emptyset$ is K-trivial.

It's not hard to see that their definition coincides with ours.

1. Basics

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3. A local condition characterizing K-trivials

The local condition

As usual we fix a computable metric space $\mathcal{M}$.

Let the letters p, q range over special points in $\mathcal{M}$.

We write $K(u, v)$ for $K(\langle u, v\rangle)$, the complexity of the ordered pair.

Definition

We say that $x \in M$ is locally K-trivial via b if

$$\forall n\, \exists p \text{ special } \left[d(x,p) \leq 2^{-n} \text{ and } K(p,n) \leq K(n) + b\right].$$

- Given a set $A \subseteq \mathbb{N}$, from the tuple $A\upharpoonright_n$ we can determine n. But from an approximation p we cannot (in general) determine the "intended distance" to x.
- So it seems this definition is the appropriate analog of the usual definition $K(A\upharpoonright_n) \leq K(n) + O(1)$ in Cantor space, not the less stringent condition where we write $K(p) \leq K(n) + b$.

The less stringent local condition fails even in Cantor space

In Cantor space $\{0,1\}^{\mathbb{N}}$, the special points are the infinite sequences of bits that are eventually 0.

Theorem

There is a Turing complete set $A \in \{0,1\}^{\mathbb{N}}$ such that A has c.e. complement, and

$$\forall n\, \exists p \text{ special } [d(A,p) \leq 2^{-n} \wedge K(p) \leq K(n) + b].$$

And a K-trivial set is never Turing complete. Thus, we indeed need to write $K(p,n) \leq K(n) + b$ in our generalization of K-triviality.

Equivalence with K-triviality

The local condition seems to be weaker than the definition via Cauchy names, because we only require that pairs $\langle p, n \rangle$ be compressible, not the whole tuple of special points for distances down to 2^{-n}. But, surprisingly:

Theorem

$x \in M$ is K-trivial $\Longleftrightarrow$ x satisfies the local condition for some b:

$$\forall n \, \exists p \text{ special } \left[d(x,p) \leq 2^{-n} \text{ and } K(p,n) \leq K(n) + b \right].$$

Proof idea.

$\Longrightarrow$

This is easy: let f be a K-trivial Cauchy name for x. Given n, let $p = f(n)$. Then

$$K(p,n) \leq K(f \restriction_{n+1}) + O(1) \leq K(n) + O(1).$$

Equivalence with K-triviality

Theorem (again)

$x \in M$ is K-trivial $\Longleftrightarrow \exists b$

$$\forall n \, \exists p \text{ special } \left[d(x,p) \le 2^{-n} \wedge K(p,n) \le K(n) + b \right].$$

Proof idea, $\Longleftarrow$ Recall that we identify special points with numbers.

- Fix a Solovay function h (i.e., h is computable, $\forall n \, K(n) \le h(n)$, and $K(n) = h(n)$ for infinitely many n).
- Consider the infinite computably enumerable tree

$$T = \{(p_1, \ldots, p_r) \colon \; d(p_i, p_{i+1}) \le 2^{-i-1} \text{ and } K(p_i, i) \le h(i) + b\}$$

for all i (where it makes sense).

$$T = \{(p_1, \ldots, p_r) \colon \ (p_i, p_{i+1}) \le 2^{-i-1} \text{ and } K(p_i, i) \le h(i) + b\}.$$

- Enumerate a c.e. tree G of initial segments of Cauchy names that are K-trivial for the same constant.
- When a new string η enters the given tree T, pick the largest v such that $\eta(v-1)$ is already the last entry of some string $\rho \in G$.
- By maximality of v there must be short "unused" descriptions of pairs $\langle \eta(k), k \rangle$ for $v \le k < r$. We use these to be able to put ρ concatenated with the rest of η on G.
- G is an infinite, finitely branching tree. Hence it has a path, which is a Cauchy name for x. It is K-trivial by a theorem on Solovay functions by Bienvenu, Merkle and Nies (2010).

Algorithmic information theory characterization of computable points

Theorem (variant for C)

$x \in M$ *has a* C*-trivial Cauchy name* $\iff \exists b$
$\forall n \exists p$ *special* $\left[d(x,p) \leq 2^{-n} \wedge C(p,n) \leq C(n) + b\right]$.

Note that a function $\mathbb{N} \to \mathbb{N}$ is C-trivial iff it is computable. So this characterizes computable points locally.

Future directions

- Characterize the K-trivial points in computable Banach spaces, such as $\mathcal{C}[0,1]$, ℓ_2, ...
- Study K-trivial compact sets in Cantor space and other computable metric spaces.
- A point $x \in \mathcal{M}$ is called incompressible in approximation if all special points close to x are incompressible:
$$\exists b \in \mathbb{N}\, \forall p\, [d(z,p) \geq 2^{-K(p)-b}].$$
This notion is opposite to K-triviality. In the unit interval it coincides with Martin-Löf randomness. We showed some closure properties. Relate incompressibility to K-triviality in $\mathcal{M}$, possibly via relative computability in $\mathcal{M}$.

Summary

- K-triviality can be defined in general computable metric spaces.
- If the space is perfect they can be non-computable.
- Reasonable examples in spaces like ℓ_2 and $\mathbb{K}(\{0,1\}^{\mathbb{N}})$.
- Preserved under computable maps, hence independent of the particular computable structure.
- The definition via having a K-trivial Cauchy name is equivalent to the apparently weaker local definition

$$\forall n \, \exists p \text{ special } \left[d(x,p) \leq 2^{-n} \wedge K(p,n) \leq K(n) + b\right].$$

References

- these and other slides, on Nies' web page.
- Paper by Melnikov and Nies, Proc. AMS, in press
- Nies' book for background on K-triviality.