Analysis and Randomness in Auckland: Introduction

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"Almost everywhere" theorems

Several theorems in real analysis assert a property for almost every real. Often they state that a function of a certain type is well-behaved at almost every input.

Theorem (Lebesgue, 1904)

Let $f : [0, 1] \to \mathbb{R}$ *be non-decreasing. Then the derivative* $f'(z)$ *exists for almost every real* z*, that is, with (uniform) probability 1.*

Ergodic theory

Definition

A measurable operator T on a probability space $(M, \mathcal{A}, \mu)$ is called ergodic if for each $X \in \mathcal{A}$,

- $\mu(X) = \mu(T^{-1}(X))$, and
- $T^{-1}(X) = X$ implies $\mu(X) = 0$ or $\mu(X) = 1$.

Examples: Cantor space $\{0,1\}^{\mathbb{N}}$, T is shift map.
Unit interval, $T(x) =$ fractional part of $x + \alpha$, where $\alpha > 0$ is irrational.

Theorem (Ergodic Theorem; Birkhoff, 1932)

Let $f \in L^1(\mu)$. Then for almost every $x \in M$, the "time average" $\frac{1}{N}\sum_{i=0}^{N-1} f \circ T^i(x)$ converges to the "space average" $\int f d\mu$.

Connection to computable analysis, and algorithmic randomness

For each theorem of this kind:

- Find a framework in which the given objects are computable.
- Now the "almost everywhere" property may correspond to an algorithmic randomness notion. Try to figure out which one.

For Lebesgue's theorem, the notion is "computable randomness", which is based on effective betting strategies.

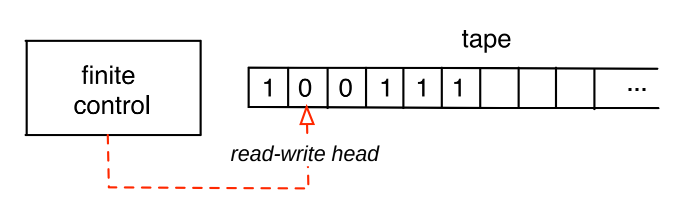
For the Ergodic Theorem, the notion can be Schnorr randomness, or Martin-Löf randomness (both are defined shortly). It depends on the type of effectiveness required for the function f.

Part I

Brief introduction to computability

Basics of computability theory 1

A Turing machine in action looks like this:



The finite control holds a Turing program.

A function $F\colon \mathbb{N} \to \mathbb{N}$ is called computable if there is a Turing program which, beginning with n in binary on the tape, ends with $F(n)$ in binary on the tape:

$$n \longrightarrow \boxed{\text{Turing program}} \longrightarrow F(n)$$

Basics of computability theory 2

A function $\psi\colon \mathbb{N} \to \mathbb{N}$ is partial computable if there is a Turing program which, with n on the input tape, outputs $\psi(n)$ if defined, and loops forever otherwise.

$n \longrightarrow$ Turing program $\longrightarrow \psi(n)$ if $\psi(n)$ is defined

$n \longrightarrow$ Turing program ☹ if $\psi(n)$ is undefined

We say that $A \subseteq \mathbb{N}$ is computably enumerable (c.e.) if one can effectively enumerate the elements of A in some order.

Basics of computability theory 3

$(W_e)_{e \in \mathbb{N}}$ is an effective listing of all the computably enumerable sets.

The halting problem is a universal computably enumerable set:

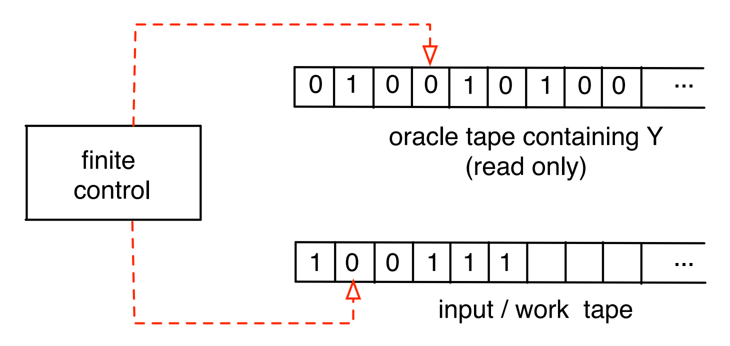
$$\mathcal{H} = \{\langle x, e \rangle \colon \ x \in W_e\}.$$

Basics of computability theory 4

For sets $X, Y \subseteq \mathbb{N}$, we write

$$X \leq_{\mathrm{T}} Y$$

(X is Turing below Y) if an "oracle" Turing machine can compute X asking queries to Y on its oracle tape.



Part II

Brief introduction to computable analysis

Computable reals

In the definition of computable functions, $\mathbb{N}$ can be replaced by domains that are effectively encoded by natural numbers, such as the rationals $\mathbb{Q}$.

- A real $r \in \mathbb{R}$ is computable if there is a computable sequence $(q_n)_{n\in\mathbb{N}}$ of rational numbers such that $|r - q_n| < 2^{-n-1}$ for each n.
- $\sqrt{2}, \pi, e$ are computable reals
- To define a non-computable real, one needs computability theory. Examples of such reals are
  - $\sum_{n\in\mathcal{H}} 2^{-n}$, where $\mathcal{H}$ is the halting problem
  - Chaitin's Ω.

Computable functions on the unit interval

Definition

We say that a function $f\colon [0,1] \to \mathbb{R}$ is computable if

(a) For each rational $q \in [0,1]$, the real $f(q)$ is computable uniformly in q.

(b) f is effectively uniformly continuous: for input a rational $\epsilon > 0$ we can compute a rational $\delta > 0$ such that

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \epsilon.$$

In general, the condition (a) by itself is too weak. However,

- if a nondecreasing function f satisfies (a) and is continuous, then it is already computable.
- For a Lipschitz function f, (a) is also sufficient.

For instance, the functions e^x, and $\sqrt{x}$, and $\sin x$ are computable.

Part III

An even briefer introduction to algorithmic randomness

Randomness via effective Vitali covers

Let $(G_k)_{k \in \mathbb{N}}$ be a computable sequence of rational open intervals with $|G_k| \to 0$.
The set of points Vitali covered by $(G_k)_{k \in \mathbb{N}}$ is

$$\mathcal{V}(G_k)_{k \in \mathbb{N}} = \{z \colon z \text{ is in infinitely many } G_k\text{'s}\}.$$

Martin-Löf and Schnorr randomness also can be defined via effective Vitali covers.

- Martin-Löf random: not in any set $\mathcal{V}(G_k)_{k \in \mathbb{N}}$ where $\sum_k |G_k| < \infty$
- Schnorr random: not in any set $\mathcal{V}(G_k)_{k \in \mathbb{N}}$ where $\sum_k |G_k|$ is a computable real.

Part IV

Back to Lebesgue and Birkhoff

Effective form of Lebesgue's theorem

Theorem (Brattka, Miller, N; submitted)

Let $f : [0,1] \to \mathbb{R}$ be non-decreasing and computable. Then $f'(z)$ exists for every computably random real z.

If we merely know that f has bounded variation, then $f'(z)$ exists for each Martin-Löf random real z (Demuth, 1975).

Effective forms of the ergodic theorem

Theorem (Galatolo, Hoyrup, Rojas 2010)

Let $T\colon [0,1] \to [0,1]$ and $f\colon [0,1] \to \mathbb{R}$ be computable. Suppose T is ergodic. Then for every Schnorr random $x \in [0,1]$,

$$\frac{1}{N}\sum_{i=0}^{N-1} f \circ T^i(x) \text{ converges to } \int f d\mu.$$

Note that $\int f d\mu$ is a computable real.
They obtained the result actually in the much more general setting of a "computable probability space".

Theorem (4+5 authors)

If f is merely "upper semicomputable" then the conclusion holds for each Martin-Löf random x.

Part V

Can computable analysis help to understand algorithmic randomness?

Characterizing Turing complete Martin-Löfrandom reals by density

Let E be a subset of $[0,1]$. The dyadic lower density of a real $z \in E$ is

$$\liminf_{|J| \to 0} \frac{\lambda(J \cap E)}{|J|},$$

where J ranges over intervals with dyadic rational endpoints that contain z, and λ is uniform (Lebesgue) measure.

Theorem (Bienvenu, Hölzl, Miller, N, STACS 2012)

Let z be a Martin-Löf random real. Then

z is Turing above the halting problem $\Leftrightarrow$

z is a point of lower density 0 *in some effectively closed set* $E \subseteq [0,1]$.

Using this to solve a long-standing open question

Reals in $[0, 1]$ are identified with subsets of $\mathbb{N}$ via the binary expansion.
K-trivial sets are far from random in a specific sense.
The analysis viewpoint of Turing completeness for Martin-Löf random sets leads to the following result:

K-trivial sets don't help ML-random sets to compute the halting problem.

More specifically:

Theorem (Day and Miller, recent)

Let $A \subseteq \mathbb{N}$ be K-trivial.
Suppose $Z \subseteq \mathbb{N}$ is a Martin-Löf random set such that Z and A together compute the halting problem.
Then already Z computes the halting problem.