Measuring the complexity of Δ^0_2 sets via their changes

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Δ^0_2 sets

In computability theory one studies the complexity of sets Z of natural numbers.

A good arena, not too far from the computable sets, is the class of Δ^0_2 sets, that is, the sets Turing below the Halting problem $\emptyset'$.

For, they can still be approximated in a computable way via the Limit Lemma.

Computable approximations of sets

Theorem (Shoenfield Limit Lemma,1959)

Z is Turing below the halting problem $\emptyset'$ $\Longleftrightarrow$
there is a computable function $g\colon \mathbb{N}\times\mathbb{N}\to\{0,1\}$ such that

$$Z(x) = \lim_s g(x,s)$$

for each $x \in \mathbb{N}$.

We will write Z_s for $\{x\colon\ g(x,s)=1\}$. The sequence $(Z_s)_{s\in\mathbb{N}}$ is called a computable approximation of Z.

Main idea

Study the complexity of a Δ^0_2 set Z by quantifying the amount of changes needed in a computable approximation $(Z_s)_{s\in\mathbb{N}}$.

First we will do this for random sets.

Then we do it for computably enumerable (c.e.) sets.

Thereafter we will relate the two cases.

Martin-Löf randomness

Our central algorithmic randomness notion is the one of Martin-Löf. It has many equivalent definitions. Here is one.

Z is Martin-Löf random $\Leftrightarrow$

> for every computable sequence $(\sigma_i)_{i\in\mathbb{N}}$ of binary strings with $\sum_i 2^{-|\sigma_i|} < \infty$, there are only finitely many i such that σ_i is an initial segment of Z.

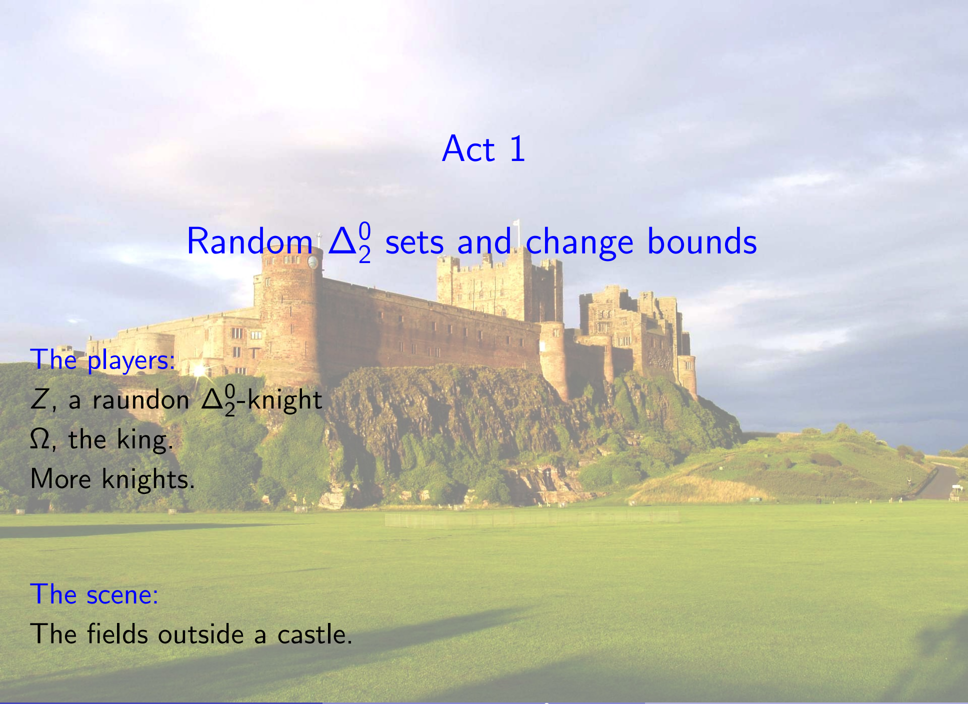
Note that $\lim_i 2^{-|\sigma_i|} = 0$, so this means that we cannot "Vitali cover" Z, viewed as a real number, with the collection of dyadic intervals corresponding to $(\sigma_i)_{i\in\mathbb{N}}$.

Left-c.e. sets

We consider a special type of Δ^0_2 set. We say that $Z \subseteq \mathbb{N}$ is left-c.e. if it has a computable approximation $(Z_s)_{s \in \mathbb{N}}$ such that $Z_s \leq_{lex} Z_{s+1}$ in the lex ordering.

For instance, let Ω be the halting probability of a universal prefix-free machine $\mathbb{U}$.
Then Ω is left-c.e.: Ω_s is the measure of $\mathbb{U}$-descriptions σ where the computation $\mathbb{U}(\sigma)$ has converged by stage s. This is a dyadic rational, identified with a binary string.

Ω is a left-c.e. random set.



Act 1

Random Δ^0_2 sets and change bounds

The players:

Z, a raundon Δ^0_2-knight

Ω, the king.

More knights.

The scene:

The fields outside a castle.

Counting the changes of initial segments

Definition

Let $g\colon \mathbb{N} \to \mathbb{N}$. We say that a Δ^0_2 set Z is a

g-change set

if it has a computable approximation $(Z_s)_{s\in\mathbb{N}}$ such that an initial segment $Z_s \upharpoonright_n$ changes at most $g(n)$ times.

We also say that Z is g-computably approximable, or g-c.a. To be ω-c.a. means to be g-c.a. for some computable g.

Example

Every left-c.e. set is a g-change set for some $g = o(2^n)$.

Proof: If $Z\upharpoonright_k$ is stable by stage t, then for $n \geq k + t + 1$, $Z\upharpoonright_n$ changes at most 2^{n-k} times.

Lower bounds for changes of raundon Δ^0_2 knights

Proposition (Figueira, Hirschfeldt, Miller, Ng and Nies, 2011)

Let Z be a random Δ^0_2 set.
Let $q : \mathbb{N} \to \mathbb{R}^+$ be computable and nonincreasing.
If Z is a $\lfloor q(n)2^n \rfloor$-change set then $\lim_n q(n) > 0$.

For instance, let $q(n) = 1/\log\log n$. Then $\lim_n q(n) = 0$. Thus:

Example

No ML-random set is a $\lfloor 2^n/\log\log n \rfloor$-change set.

As a consequence, for the number of initial segment changes for Ω, the $o(2^n)$ upper bound is not far below 2^n.

Sir Thomas Malory, Le Morte D'Arthur (1483)

Chapter IX: How Sir Tor rode after the knight with the brachet, and of his adventure by the way.

> *(...) And anon the knight yielded him to his mercy. But, sir, I have a fellow in yonder pavilion that will have ado with you anon. He shall be welcome, said Sir Tor. Then was he ware of another knight coming with great raundon, and each of them dressed to other, that marvel it was to see; but the knight smote Sir Tor a great stroke in midst of the shield that his spear all to-shivered. And Sir Tor smote him through the shield below of the shield that it went through the cost of the knight, but the stroke slew him not. (...)*

The Old French noun "raundon", great speed, is derived from "randir", to gallop. It has been used in English since the 14th century. Metaphorically, "raundon" also meant 'impetuousity'. (OED)

Sir Thomas Malory's thesis

Let Z be a Martin-Löf random Δ^0_2 set.

Z gets more raundon

$\Leftrightarrow$

Z needs more changes.

Was Malory right?

Enhancing Martin-Löf randomness

Let Z be a Martin-Löf random set. The randomness enhancement principle states that

Z is computational less complex $\iff$ Z is more random

This was observed first for randomness notions not compatible with Δ^0_2.

Example (Hirschfeldt, Miller '06 / N, Terwijn, Stephan '05)

- Z and $\emptyset'$ form a minimal pair $\iff$ Z is weakly 2-random
- Z is low for Ω $\iff$ Z is 2-random.

The following later result also affects a Δ^0_2 ML-random set Z.

Example (Franklin, Ng 10)

Z is Turing incomplete $\iff$ Z is difference random.

For random Δ^0_2 sets, less complex means more changes (1)

Randomness enhancement together with Malory's thesis would imply that for a Martin-Löf random Δ^0_2 set Z,

Z is computational less complex $\Longleftrightarrow$ Z needs more changes.

To give evidence for this, first we consider random Δ^0_2 sets that are complex. This should mean that they need few changes.

Example

Ω is Turing complete.
Its rate of change is $o(2^n)$, which is at the lower end of the scale.

For each incomputable c.e. A there is a Δ^0_2 random Z not above A. However, the ω-c.a. random sets are "jointly" complex, because:

Example (Hirschfeldt, Miller, 2006)

There is an incomputable c.e. set below all ω-c.a. ML-random sets.

For random Δ^0_2 sets, less complex means more changes (2)

Next we consider random Δ^0_2 sets that are not complex. This should mean that they need a lot of changes.

Recall that a set $Z \subseteq \mathbb{N}$ is low if $Z' \leq_{\mathrm{T}} \emptyset'$, and superlow if $Z' \leq_{\mathrm{tt}} \emptyset'$.

Theorem (Figueira, Hirschfeldt, Miller, Ng, Ni 2010)

Suppose that a Martin-Löf random set Z is superlow.
Then Z is not an $O(2^n)$ change set.

In contrast:

Theorem (Figueira et al. 2011)

There is a low Martin-Löf random set Z with $o(2^n)$ changes.

For random Δ^0_2 sets, less complex means more changes (3)

We last result also gives contrary evidence.

Theorem (Figueira et al. 2011)

There is a low Martin-Löf random set Z with $o(2^n)$ changes.

It says that Z has a rate of change similar to the one of Ω.

We would need a fine analysis of change bounds in $o(2^n)$ to differentiate between Ω and low random sets.

Act 2

Computably enumerable sets and cost functions

The players:

A, an abject Δ^0_2 peasant.

The king's tax collector.

Peasant folk.

The scene:

A village.

Cost functions

The king issues a tax law (cost function) **c**.

Consider a computable approximation $(A_s)_{s \in \mathbb{N}}$ of a Δ^0_2 peasant A.
Suppose that on day s, the number x is least such that $A_s(x)$ changes.
Then the tax the peasant pays is $\mathbf{c}(x, s) \in \mathbb{Q}^+$.

Definition

We say a Δ^0_2 set A obeys a cost function **c** if A has a computable approximation such that the total tax is finite.

Properties of **c** we require:

- computable,
- nondecreasing in s,
- nonincreasing in x.

Each fair tax law can be obeyed without being taxed to death (where death = computable)

Let $\mathbf{c}^*(x) = \sup_s \mathbf{c}(x, s)$. We say that $\mathbf{c}$ has the limit condition if $\lim_x \mathbf{c}^*(x) = 0$. (Fair tax law.)

Proposition (Existence; DHNS 03)

Suppose a cost function $\mathbf{c}$ *has the limit condition.*
Then there is a promptly simple set A *obeying* $\mathbf{c}$.

Proposition (C.e. cover; Nies' 2009 book)

Suppose a Δ^0_2 *set* A *obeys a cost function* $\mathbf{c}$.
Then there is a computably enumerable set $D \geq_{tt} A$ *such that* D *also obeys* $\mathbf{c}$.

So, for studying obedience to a single cost function we can pretty much stay with the c.e. sets.

A cost function characterizing K-trivality

Recall that A is K-trivial if for some b, $\forall n\, K(A\upharpoonright_n) \leq K(n) + b$.
This means far from random (using the Levin-Schnorr theorem).

Let $\mathbf{c}^{\langle\Omega\rangle}(x, s) = \Omega_s - \Omega_x$, the amount Ω increases from x to s.

The following characterizes the K-trivial peasants as the ones that obey the king's tax law $\mathbf{c}^{\langle\Omega\rangle}$.

Theorem (N, 2005; 2010)

$$A \text{ is } K\text{-trivial} \Leftrightarrow A \text{ obeys } \mathbf{c}^{\langle\Omega\rangle}.$$

$\Leftarrow$ is not hard. $\Rightarrow$ is also not hard for c.e. A, but needs the so-called golden run method in general. (The 2005 proof was for the cost function $\mathbf{c}_{\mathcal{K}}$.)

Corollary

Every K-trivial set has a computably enumerable K-trivial above.

For c.e. sets, less complex means fewer changes (1)

Recall that for a random Δ^0_2 set Z, the paradigm was

> Z is computational less complex
>
> $\Longleftrightarrow$
>
> Z needs more changes.

For c.e. sets A, the paradigm contrasts with the one for random Δ^0_2 sets.

> A is computational less complex
>
> $\Longleftrightarrow$
>
> A obeys stricter cost functions.

For c.e. sets, less complex means fewer changes (2)

We give evidence for this. It works in fact for left-c.e. sets.

Evidence I. The case of Ω.

Example

(a) The left-c.e. set Ω is Turing complete.

(b) It obeys no reasonable cost function.

For statement (b) we use:

Fact

If $\mathbf{c}(x, s) \geq 2^{-x}$ for all x, s, then no random Δ^0_2 set obeys $\mathbf{c}$.

Evidence II. Bickford and Mills (1982) introduced superlowness of a set A, namely, $A' \leq_{\mathrm{tt}} \emptyset'$. They called these sets abject.

Theorem (N 2005)

Each K-trivial set is superlow. Thus, obeying $\mathbf{c}^{\langle\Omega\rangle}$ implies superlowness.

For c.e. sets, less complex means fewer changes (3)

Evidence III. Let J^A be a universal partial computable functional with oracle A. Strong jump traceability (Figueira, N, Stephan 2008) is a lowness property of A saying that the possible values of J^A are very limited: they are contained in tiny uniformly c.e. sets T_x.

A cost function is called benign if the number of disjoint increments by 2^{-k} is computably bounded in k. For instance, $\mathbf{c}^{\langle\Omega\rangle}$ is benign via $k \to 2^k$.

Theorem (Greenberg and N, 2010)

Let A be c.e. Then
A is strongly jump traceable $\Leftrightarrow$ A obeys each benign cost function.

Thus, peasants get poorer when they obey stricter tax laws.

Act 3

Computably enumerable sets below random Δ^0_2 sets

The players:

Z, a raundon Δ^0_2-knight

A, an abject c.e. peasant.

Knights, peasants.

The scene:

A forest between village and castle. Night.

Interaction between knights and peasants

We now consider the situation that

$$A \leq_T Z,$$

where A is a c.e. peasant and Z is a raundon Δ^0_2 knight.

We will see that

the more Z is allowed to change (in the sense of initial segments),

the less A can change (in the sense of cost functions).

This is in line with the paradigms of Act 1 and 2:
Z changes more means Z is computationally less complex.
So the set $A \leq_T Z$ is less complex as well, and hence can change less.

The knight and his subjects

The following classical theorem says that every raundon Δ^0_2 knight has c.e. peasants subject to him.

Theorem (Kučera 1986)

Let Z be a random Δ^0_2 set.
Then there is a c.e. incomputable set A such that $A \leq_T Z$.

In general, such an A is very poor.

Theorem (Hirschfeldt, N, Stephan 2005)

If Z is Turing incomplete, then A is necessarily K-trivial.

Greenberg and N (2010) have given a cost function proof of Kučera's theorem: A is a set obeying a certain cost function c_Z associated with a computable approximation of Z.

Subjects can be arbitrarily poor

There are Δ^0_2 knights Z such that a c.e. peasant subject to Z is arbitrarily poor. In fact, instead of ML-randomness of Z we can take membership in any non-empty Π^0_1 class.

Theorem (N, in preparation)

Let $\mathcal{P}$ be a non-empty Π^0_1 class. Let $\mathbf{c}$ be a cost function with the limit condition.
Then there is a Δ^0_2 set $Z \in \mathcal{P}$ such that every c.e. set $A \leq_T Z$ obeys $\mathbf{c}$.

- In the construction, the more restrictive $\mathbf{c}$, the more Z has to change.
- If $\mathbf{c}$ is benign (not too restrictive), then Z is ω-c.a.
- This is an instance of the principle above that more changes of Z mean fewer changes of $A \leq_{\mathrm{T}} Z$.

A very low random Δ^0_2 set

It is not hard to define a cost function $\mathbf{c}$ such that every c.e. set A obeying $\mathbf{c}$ is strongly jump traceable. Let $\mathcal{P}$ be a Π^0_1 class of randoms. Then we re-obtain the following:

Theorem (Greenberg, 2009)

There is a random Δ^0_2 set Z such that every c.e. set A Turing below Z is strongly jump traceable.

History:

- Greenberg built such a Z directly.
- Thereafter, Kučera and N (2009) showed that Demuth randomness of Z does the job.
- The extension to Π^0_1 classes shows that randomness isn't really necessary here. For instance, we can also take a PA complete Z.

References

- These and other slides, on Nies' web page.
- Figueira, Hirschfeldt, Miller, Ng, N. *Counting the changes of random Δ^0_2 sets.* Logic and Computation, in press.
- Greenberg and N. *Benign cost functions and lowness properties.* J. Symb. Logic 76, Issue 1 (2011)
- N. *Calculus of cost functions.* In preparation.
- My book "Computability and Randomness" for background. Paperback version with typos corrected will come out early 2012.

Exeunt omnes.