Turning a non-Archimedean group into a countable structure

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Discretization

The process of turning uncountable topological structures taken from a given class into countable, discrete structures will be called discretization.

$$\text{Topological isomorphism} \leftrightarrow \text{plain isomorphism}$$

Left side is what we are interested in; right side can be easier to study.

The process of discretization is usually Borel. Two examples.

- Stone duality: separable Stone spaces $\rightsquigarrow$ countable BAs.
- Pontryagin: profinite abelian groups $\rightsquigarrow$ countable torsion abelian groups.

I will discuss this for natural Borel classes of non-Archimedean groups.

The plan

I. Introduce non-Archimedean groups, and list the main Borel classes for this talk

II. Discretize groups in a suitable Borel class via “meet groupoids”

III. Apply an algorithmic version of this process to understand computably t.d.l.c. groups. (T.d.l.c. = the locally compact nA groups.)

I. What is a non-Archimedean group G ?

Three definitions that are equivalent up to group homeomorphism.

(M) Definition for model theorists:

$G = \mathrm{Aut}(M)$ where M is a countable structure. G is viewed as a topological group with the topology of pointwise convergence.

($\longrightarrow$ many examples)

(T) Definition for people in topological groups:

G is a Polish group such that has a countable neigbourhood basis of the identity consisting of open subgroups.

($\longrightarrow$ nice abstract definition)

(P) Definition for people in permutation groups:

G is a closed subgroup of $\mathrm{Sym}(\mathbb{N})$.

($\longrightarrow$ concrete, amenable to algorithmic study)

Why are those definitions of nA all equivalent?

(M) $\to$ (P) If $G = \mathrm{Aut}(M)$, then clearly G is a closed subgroup of $\mathrm{Sym}(M)$; we may assume the domain of M is $\mathbb{N}$.

(P) $\to$ (M) Let Can_G be the "canonical structure" for G. It has n-ary relations for the n-orbits of the acton of G on $\mathbb{N}$. Then $G = \overline{G} = \mathrm{Aut}(\mathrm{Can}_G)$.

(T) $\to$ (P) G acts faithfully from the left on the (countable) set of left cosets of subgroups in the nbhd basis.

(P) $\to$ (T) Let U_n be the pointwise stabiliser of $\{0, \ldots, n\}$. $(U_n)_{n\in\mathbb{N}}$ is nbhd basis.

The name was borrowed from theory of valued fields, because the valuation on non-Archimedean fields such as $\mathbb{Q}_p$ yields an ultrametric, and hence a totally disconnected topology.

Borel classes of non-Archimedean groups

The nA groups form a Borel subspace of the Effros space of closed subsets of $\mathrm{Sym}(\mathbb{N})$. The following classes of nA groups are Borel, and invariant under conjugation.

- Oligomorphic: G has only finitely many n-orbits, for each n. (Equivalently, the structure Can_G is ω-categorical).
  Examples: $\mathrm{Sym}(\mathbb{N})$, $\mathrm{Aut}(\mathbb{Q}, <)$, $\mathrm{Aut}(R)$, $\ldots$
- Compact (profinite): each 1-orbit is finite. $\mathrm{Aut}(\overline{\mathbb{Q}}, +, \times)$
- Locally compact: $\mathrm{Aut}(M, \overline{c})$ is compact for some $\overline{c} \in M^{<\omega}$.
  These are called t.d.l.c groups; their study is its own area. Examples: $\mathrm{Aut}(T_d)$, where $d \geq 3$ and T_d is the connected acyclic graph with each vertex of degree d.

II. Meet groupoids of groups in a Borel discretizable class

Natural nbhd basis of 1?

Recall that a Polish group G is nA iff there is a countable nbhd basis S_G of 1_G consisting of open subgroups U. Let $\mathcal{C}$ be a Borel class of nA groups.

To qualify as "natural", an assignment $G \mapsto S_G$ (for $G \in \mathcal{C}$) should be **Borel**, and **invariant** under topological isomorphisms $p\colon G \to H$ of groups in $\mathcal{C}$:

$$U \in S_G \leftrightarrow p(U) \in S_H, \text{ for each } U \leq_o G.$$

Which of the classes above have such an assignment?

- $\mathcal{C} =$ profinite, $\mathcal{C} =$ oligomorphic: take as S_G **all** open subgroups.
- $\mathcal{C} =$ locally compact: $S_G =$ the **compact** open subgroups. S_G is countable, and forms a nbhd basis of 1_G by van Dantzig.
- How about $\mathcal{C} =$ procountable?? **NO**...

Borel discretizable classes, and meet groupoids

Definition

We say that a Borel class $\mathcal{C}$ of nA groups is Borel discretizable if

- there is a Borel assignment $G \mapsto S_G$ (for $G \in \mathcal{C}$), where S_G is a countable nbhd basis of 1_G consisting of open subgroups, s.t.
- for each $G, H \in \mathcal{C}$ and $p \colon G \cong_{\text{top}} H$ we have $U \in S_G \leftrightarrow p(U) \in S_H$.

Definition (Meet groupoid $\mathcal{W}(G)$)

For $G \in \mathcal{C}$ let $\mathcal{W}(G)$ be the countable structure with domain the cosets of groups in S_G, together with $\emptyset$, and operations

- $\cap$ (total) and
- $\cdot$ the usual product in group where $A \cdot B$ is defined when A is a left coset of $U \in S_G$ and B a right coset of U.

Some remarks on the definition of $\mathcal{W}(G)$

Definition (Recall)

For $G \in \mathcal{C}$ let $\mathcal{W}(G)$ be the countable structure with domain the cosets of groups in S_G, together with $\emptyset$, and operations $\cap$, and $\cdot$ where $A \cdot B$ is defined when A is a left coset of U and B a right coset of U.

- $\mathcal{W}(G)$ is countable because S_G is, and each open subgroup has countable index.
- S_G is closed under conjugation, and $Ua = a(a^{-1}Ua)$. So

  $$\text{“left cosets of groups in } S_G = \text{ right cosets of groups in } S_G\text{”}.$$

- $aU \cdot Ub$ is defined in $\mathcal{W}(G)$ because $aUUb = aUb = \underbrace{aUa^{-1}}_{\in S_G} ab = ab \underbrace{b^{-1}Ub}_{\in S_G}$.
- $\cap$ is defined in $\mathcal{W}(G)$ because $aU \cap bV = c(U \cap V)$ whenever $c \in aU \cap bV$.

Example: $\mathcal{W}(G)$ for $G = \mathrm{Aut}(\mathbb{Q}, <)$, and the story of $\mathcal{W}(G)$

- Open cosets have the form

$$A_\sigma = \{g \in G \colon g \succ \sigma\}$$

  where σ is a finite injection preserving the order.
- $A_\tau \cdot A_\sigma = A_{\tau \circ \sigma}$ when $\mathrm{dom}(\tau) = \mathrm{range}(\sigma)$.
- $A_\tau \cap A_\sigma = A_{\tau \cup \sigma}$ when $\tau \cup \sigma$ is an injection, else $\emptyset$.

The idea to use an algebraic structure on open cosets developed slowly:

- Kechris, N. Tent 2018: introduce "Borel discretizable" classes, mention idea
- N., Schlicht, Tent 2021: oligomorphic groups $\rightsquigarrow$ coarse groups, $\cong_{\text{olig}} \leq_B E_\infty$.
- Melnikov, N. 2026, posted 2022: t.d.l.c. groups $\rightsquigarrow$ meet groupoids, in computable setting

Turning nA groups into countable structures

Theorem (Borel category equivalence, with Schlicht, Fund. Math. 2026)

Let $\mathcal{C}$ be Borel discretizable class that is closed under (topological) isomorphism. View $\mathcal{C}$ with isomorphisms as a "Borel category". There is a **Borel category** $\mathcal{M}$ of meet groupoids with isomorphisms, and a **Borel functor** $\mathcal{G}\colon \mathcal{M} \to \mathcal{C}$, such that $\mathcal{W}$ and $\mathcal{G}$ induce an **equivalence of categories**. In particular,

$$\mathcal{G}(\mathcal{W}(G)) \cong_{\text{top}} G \text{ and } \mathcal{W}(\mathcal{G}(M)) \cong M, \text{ for each } G \in \mathcal{C}, \text{ and } M \in \mathcal{M}.$$

Proof: Given a meet groupoid M, may assume its domain is $\mathbb{N}$.
Let $\mathcal{G}(M)$ be the closed subgroup of $\mathrm{Sym}(\mathbb{N})$ consisting of the permutations p such that for each $A, B \in M$

$$p(A \cap B) = p(A) \cap p(B), \text{ and } A \cdot B \text{ defined} \to p(A \cdot B) = p(A) \cdot B.$$

[redacted]

Theorem (Recall)

Let $\mathcal{C}$ be a Borel discretizable class that is closed under isomorphism. There is a Borel category $\mathcal{M}$ of meet groupoids with isomorphism and a Borel functor $\mathcal{G}\colon \mathcal{M} \to \mathcal{C}$ such that $\mathcal{W}$ and $\mathcal{G}$ induce an equivalence of categories.

Corollary

(a) $\cong_{\mathcal{C}}$ is CCS, i.e. $\leq_B$ countable graph isomorphism. (So, the class of procountable groups is not Borel discretizable... one can't Borel pick a natural nbhd basis of 1)

(b) $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathcal{W}(G))$, and then $\mathrm{Aut}(G)$ has a Polish topology that makes its action on G continuous. (Done first for oligomorphic groups, with Paolini.)

Such a topology on $\mathrm{Aut}(G)$ is unique. Our direct proof of (b) uses an embedding of $\mathrm{Aut}(G)$ as the normaliser of $\widehat{G}$, where $\widehat{G} \leq_c \mathrm{Sym}(\mathbb{N})$ is a suitable copy of G.

Locally Roelcke precompact groups

Definition

$G \leq_c \mathrm{Sym}(\mathbb{N})$ is Roelcke precompact if each open subgroup U has only finitely many double cosets (things of the form UxU). G is locally Roelcke precompact if it has a Roelcke precompact open subgroup. E.g., $\mathrm{Aut}(T_\infty)$ is locally R.p., and not t.d.l.c.

Diagram of some Borel discretizable classes with inclusions

$$\begin{tikzcd}[arrows=-]
& \text{locally Roelcke precompact} \arrow[dl, "\subset"'] \arrow[dr, "\supset"] & \\
\text{Roelcke precompact} \arrow[d, "\cup"'] \arrow[dr, "\supset"'] & & \text{locally compact} \arrow[dl, "\subset"'] \arrow[d, "\cup"'] \\
\text{oligomorphic} & \text{compact} & \text{discrete}
\end{tikzcd}$$

$\cong$ on oligomorphic groups is essentially countable (NST, 2021)

- The general category equivalence only works if the class $\mathcal{C}$ is closed under $\cong$.
- $\mathcal{C}$ = oligomorphic groups is not closed. In N., Schlicht and Tent 2021, we used a $\mathcal{G}$ ensuring that $\mathcal{G}(\mathcal{W}(G))$ is indeed oligomorphic (based on "coarse groups").
- This shows the main result that $\cong_{\mathcal{C}}$ is essentially countable, using:

Theorem (Hjorth and Kechris, APAL 1997, Th. 3.8)

Let $\mathcal{D}$ be a Borel class of structures with domain $\mathbb{N}$. Suppose that $\cong_{\mathcal{D}} \leq_B L$ for some F_σ equivalence relation L on some Polish space Y. Then $\cong_{\mathcal{D}}$ is essentially countable.

For our proof, $\mathcal{D}$ is the isomorphism closure of $\{\mathcal{W}(G) \colon G \in \mathcal{C}\}$. The F_σ relation L is bi-interpretability on ω-categorical structures. For $R, S \in\ +D$, we have by Conquand

$$R \cong S \iff \mathcal{G}(R) \cong \mathcal{G}(S) \iff \mathrm{Can}_{\mathcal{G}(R)} \text{ is bi-interpretable with } \mathrm{Can}_{\mathcal{G}(S)}.$$

Smoothness for oligomorphic groups without algebraicity

Theorem (with Paolini, BPAM 2026)

Suppose for a Borel subclass $\mathcal{E}$ of the oligomorphic groups, we have an isomorphism invariant Borel assignment $G \mapsto P_G$ (countable set of open subgroups) such that

- P_G has only finitely many conjugacy classes, and
- the finite intersections of subgroups in P_G form a nbhd basis of 1_G.

Then $\cong_{\mathcal{E}}$ is smooth.

Proof: build for $G \in \mathcal{E}$ a "trimmed-down" version of $\mathcal{C}_G$ of the meet groupoid $\mathcal{W}(G)$ that is ω-categorical; then use that isomorphism of ω-categorical structures is smooth.

Now let $\mathcal{E}$ be the class of oligomorphic groups G such that Can_G has no algebraicity. Show that $\cong_{\mathcal{E}}$ is smooth: Let P_G be the set of point stabilisers. By Rubin (1994), for $G, H \in \mathcal{E}$, each isomorphism $p \colon G \cong H$ is given by a bijection r between the canonical structures for G and H in sense that $p(\alpha) = r \circ \alpha \circ r^{-1}$.

III. Computably t.d.l.c. groups

(with Melnikov, JSL 2026)

Computable version of category equivalence theorem?

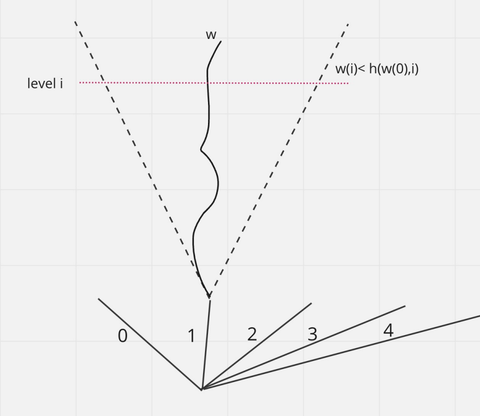
- The equivalence of categories $\mathcal{C}$ (groups) and $\mathcal{M}$ (meet groupoids) is Borel.
- Some of the classes of nA groups have computable versions: discrete (Malcev, Rabin), computably profinite (Smith, La Roche). For others, such as oligomorphic, algorithmic versions wait to be developed. (Greenberg, Melnikov, N., Turetsky, PAMS 2018)
- When is there a category equivalence theorem where "everything is computable"?
- There is one for the t.d.l.c. case (and hence also the profinite) case.

Recall that a nA group is t.d.l.c. iff it has a nbhd basis of 1 consisting of compact open subgroups.

Computable Baire presentation of a t.d.l.c. group G

- The domain of such a presentation of G equals $[T]$ for a computable subtree of $\mathbb{N}^*$ without leaves such that
  - the only possible infinite branching is at the root
  - there is a computable bound $h\colon \mathbb{N} \to \mathbb{N}$ such that $w(i) \leq h(i, w(0))$ for each $w \in T$ and $i > 0$. (The tree above n is finitely branching, effectively in n.)
- The operations of G are computable.

If it exists, we say that G is computably t.d.l.c



$\mathbb{F}_p((t))$ and $\mathrm{Aut}(T_d)$ are computably t.d.l.c.

Let T be the tree of strings $\tau \in \mathbb{N}^*$ such that

- all entries, except possibly the first, are among $\{0, \dots, p-1\}$,
- $\tau(0) > 0 \to \tau(1) > 0$.

String $r\sigma \in Q$ denotes the Laurent polynomial $x^{-r} \sum_{0 \le k < |\sigma|} \sigma(k) x^k$.
One checks that addition on $[T]$ is computable. $\mathbb{Q}_p$ works in a similar way.

For $\mathrm{Aut}(T_d)$, the bottom level of T tells where a fixed vertex v of T_d goes. The k-th level says where vertices at distance k from v go.

Computably t.d.l.c. groups via meet groupoids

We have an abstract notion of meet groupoid, with axioms expressing the obvious f.o. properties of each $\mathcal{W}(G)$. The "abstract subgroups" U are the idempotents: $U \cdot U = U$. A meet groupoid $\mathcal{W}$ is called Haar computable if

(a) its domain is a computable subset $\mathbb{D}$ of $\mathbb{N}$;

(b) the groupoid and meet operations are computable; in particular, the relation $\{\langle A, B\rangle : A, B \in \mathbb{D} \wedge A \cdot B \text{ is defined}\}$ is computable;

(c) the function sending a pair of idempotents $U, V \in \mathcal{W}$ to the number of (left, say) "cosets" of $U \cap V$ in U is computable.

Definition (Computably t.d.l.c. groups via meet groupoids)

Let G be a t.d.l.c. group. We say that G is computably t.d.l.c. via a meet groupoid if $\mathcal{W}(G)$ has a Haar computable copy $\mathcal{W}$.

Computable discretization for t.d.l.c. groups

Theorem (Melnikov and N., arXiv 2022, JSL 2026)

A group G is computably t.d.l.c. via a Baire presentation $\iff$

G is computably t.d.l.c. via a meet groupoid.

From a presentation of G of one type, one can effectively obtain a presentation of G of the other type.

Proof. $\Rightarrow$. Build the copy $\mathcal{W}$ of $\mathcal{W}(G)$ on compact cosets, represented by certain finite subsets of T. Use the computable bound on the branchings in T to show that $\mathcal{W}$ is Haar computable.

$\Leftarrow$. Given M, assume the domain is $\mathbb{N}$. Build a computable tree T such that the pathsin $[T]$ denote the pairs $\langle p, p^{-1}\rangle$ where $p \in \mathcal{G}(M)$. Haar computability of M yields the branching bound.

A noncomputable scale function

By how much must conjugation by $g \in G$ "shift" every compact open subgroup? Following Willis (1994), for a t.d.l.c. group G, the scale function $s \colon G \to \mathbb{N}^+$ is

$$s(g) = \min\{|V^g \colon V \cap V^g| \colon V \text{ is a compact open subgroup}\}.$$

Theorem (Melnikov,N. 2026, with G. Willis)

There is a computable Baire presentation of a t.d.l.c. group G, based on a tree T, such that the scale function $s \colon [T] \to \mathbb{N}$ is not computable.

In fact, there is a uniformly computable sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $s(g_n) = 2$ if $n \notin \mathcal{K}$ (the halting problem), and 1 otherwise.

Two uses of meet groupoids for computably t.d.l.c. groups

Theorem (Melnikov, N., 2026, Thm. 9.9)

Let G be computably t.d.l.c. via tree T. Let N be a closed normal subgroup of G such that the subtree corresponding to N is computable. Then G/N is computably t.d.l.c.

We prove this by building a Haar computable copy of the meet groupoid of G/N.
Application: $PGL_n(\mathbb{Q}_p)$ is computably t.d.l.c. .

Theorem (Melnikov and N. 2026, Thm. 10.6)

Let p be a prime. The computably t.d.l.c. groups $\mathbb{Q}_p$ and $\mathbb{Q}_p \rtimes \mathbb{Z}$ are autostable: for any two Baire computable presentations, there is a bi-computable isomorphism.

This works by showing any two Haar computable copies of the meet groupoids are computably isomorphic, and using a transfer theorem.

Cooperation with researchers in t.d.l.c. groups

Computational Aspects of Totally Disconnected Locally Compact Groups

Michal Ferov, Stephan Tornier, and George Willis

Handbook of Visual, Experimental and Computational Mathematics: Bridges through Data, pp. 1–36, Springer Nature Switzerland, Cham, 2025

Connections of oligomorphic and t.d.l.c in computable setting?

An oligomorphic group G will be called computably oligomorphic if its canonical structure Can_G, as well as the orbit counting function, are computable. Examples such as $\mathrm{Aut}(\mathbb{Q}, <)$, $\mathrm{Aut}(R)$ are computable.

Question

How is this related to the complexity of $\mathcal{W}(G)$? How complex is the nA group $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathcal{W}(G))$?

Question

What can we say about the computational complexity of the profinite group N_G/G? How complex is the t.d.l.c. group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$?

N. and Schlicht show $\mathrm{Out}(G) \cong$ the group of self bi-interpretations of the theory of Can_G, modulo "definable homotopy". This should give some upper bound.

A. Melnikov and A. Nies.

Computably totally disconnected locally compact groups.

J. Symbolic Logic, in press. arxiv.org/pdf/2204.09878.pdf., 2024.

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Oligomorphic groups, their automorphism groups, and the complexity of their isomorphism. BPAM, in press. arxiv.org/pdf/2410.02248.

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Automorphism groups of non-Archimedean groups.

Fund. Math., in press. arxiv.org/pdf/2512.12589.pdf.

A. Nies, P. Schlicht, and K. Tent.

Coarse groups, and the isomorphism problem for oligomorphic groups.

J. Math. Log. **22** (2022), no. 01, 2150029.