Profinite groups and computability theory

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The plan

I. Profinite groups: definition and examples

II. Algorithmic presentations of profinite groups

III. Fractal dimensions of closed subgroups
(with Elvira Mayordomo, U Zaragoza)

IV. Algorithmic randomness in computable profinite groups
(with Willem Fouche, Pretoria and Matteo Vannacci, U Firenze)

I. Profinite groups

their definitions and examples

Profinite groups as inverse limits

An inverse system is a sequence $(G_n, p_n)_{n\in\mathbb{N}}$ where the G_n are finite groups, and the $p_n \colon G_{n+1} \to G_n$ are homomorphisms.

Its inverse limit is the topological group $G = \varprojlim_n (G_n, p_n)$, given up to isomorphism by the universal property from category theory.

A separable topological group is called profinite if

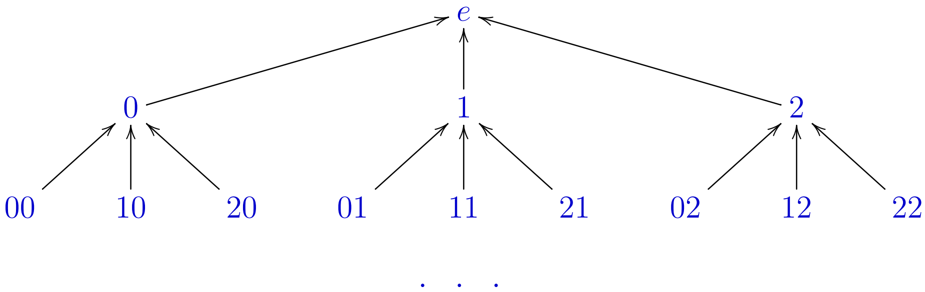
it is isomorphic to such an inverse limit.

Equivalently, the group is compact and 0-dimensional.

G will always denote a profinite group with a specified inverse system.

The inverse limit as a group on a path space (1)

An inverse system $(G_n, p_n)_{n\in\mathbb{N}}$, with G_0 trivial, yields a finitely branching rooted tree T. The n-th level consists of G_n; the predecessor relation is given by the $p_n\colon G_{n+1} \to G_n$.



The first levels of the tree for the additive group of 3-adic integers.
$G_1 = C_3$, $G_2 = C_9$, etc.

The inverse limit as a group on a path space (2)

Recall: an inverse system of groups $(G_n, p_n)_{n\in\mathbb{N}}$, with G_0 trivial, yields a finitely branching rooted tree T. The n-th level consists of G_n; the predecessor relation is given by the maps $p_n\colon G_{n+1} \to G_n$.

- As the domain of the inverse limit one can concretely take the path space $[T]$.
- Its neutral element is the path consisting of the neutral elements in the G_n's.
- The group multiplication is given by

$$f \cdot g = \bigcup_n \left[f\restriction_n \cdot_n g\restriction_n\right] \text{ for } f, g \in [T].$$

- Similarly for inverse operation.
- These operations are continuous w.r.t. the topology on $[T]$.

Profinite groups given by p-adic integers

- Let p be a prime. Let $(\mathbb{Z}_p, +) = \varprojlim_n C_{p^n}$ where C_{p^n} is the cyclic group of size p^n.

- Via the view as a tree, the elements of $\mathbb{Z}_p$ can be encoded by infinite sequences of digits in $\{0, \dots, p-1\}$, with addition via the usual carry digits.

- This is a pro-p group: all the G_n have size a power of p.

- Let $k \geq 2$. Matrix groups such as
  - upper unitriangular $\mathrm{UT}_k(\mathbb{Z}_p)$
  - special linear $\mathrm{SL}_k(\mathbb{Z}_p)$

  are profinite. This uses that $(\mathbb{Z}_p, +, \times)$ is a profinite ring.

Profinite groups given by infinite Galois extensions

For a Galois extension K/k, its Galois group $G = \mathrm{Gal}(K/k)$ consist of the automorphisms of K that fix k pointwise.

- $G = \mathrm{Gal}(K/k)$ is a profinite group:
- If $K = \bigcup_{i\in\mathbb{N}} L_i$, where $L_{i+1} \geq L_i$ and each L_i is a normal finite extension of k, then

$$G \cong \varprojlim_{i} \left(\mathrm{Gal}(L_i/k), p_i\right)$$

where $p_i(\sigma)$ is the restriction of σ to L_i.

Profinite groups from $\aleph_0$-categorical structures

Theorem (N. and Paolini, 2024, arxiv.org/pdf/2410.02248)

Let M be an $\aleph_0$-categorical structure with domain $\mathbb{N}$, and let $G = \mathrm{Aut}(M)$. Then N_G/G is profinite.

Here $N_G = \{\sigma \in \mathrm{Sym}(\mathbb{N})\colon G^\sigma = G\}$ is the normaliser of G; it coincides with $\mathrm{Aut}(\mathcal{E}_M)$ where $\mathcal{E}_M$ is the orbital structure of M. Any separable profinite group occurs (Evans and Hewitt, 1991).

Theorem (N. and Paolini, 2024)

(1) Let G as above. Then $\mathrm{Aut}(G)$ carries a natural Polish topology and $\mathrm{Inn}(G)$ is closed in it.
(2) $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ with the quotient topology is totally disconnected, locally compact (t.d.l.c.).

It is unknown whether $\mathrm{Out}(G)$ is in fact always profinite.

II. Algorithmic presentations of profinite groups

Co-c.e., and computable profinite groups

Recall: a profinite group is given by an inverse system $(G_n, p_n)_{n\in\mathbb{N}}$, where the $p_n\colon G_{n+1} \to G_n$ are homomorphisms of finite groups.

Definition (Smith, 1981; LaRoche, 1981)

A co-c.e. profinite group G is given by a computable inverse system. The group is called computable if in addition, all the p_n's are onto.

Theorem (Smith, 1981)

(i) Some co-c.e. profinite group G is not isomorphic to a computable one.
(ii) Each co-c.e. pro-p group that is topologically f.g. is computable.

Proof. **(i)** let A be a properly Σ^0_2 set of primes, and let G be a co-c.e. presentation of $\prod_{p\in A} C_p$.
(ii) uses the Frattini subgroup.

Co-c.e., and computable in terms of the tree

Recall that an inverse system $(G_n, p_n)_{n \in \mathbb{N}}$ yields a finitely branching tree T with levels consisting of the G_n.

- G is co-c.e. if
  - the tree T is computable with a computable number of successors, and
  - the group operations at each level are uniformly computable.
- G is computable if, in addition, the tree has no leaves.

Preservation properties

Smith (1981) proved preservation properties for computable profinite groups. If G is computable then

- the derived group G' is computable
- for each prime p, the group G has a computable p-Sylow subgroup (that is, a maximal pro-p closed subgroup).

Arbitrary effective tree $\rightsquigarrow$ nice effective tree

- If a topological structure for a finite functional signature σ is compact and 0-dimensional, then it has a copy with domain $[T]$ for some finitely branching tree T.
- Co-c.e. σ-structures: T is computable, with computable bound on branching, and operations computable.
- Computable σ-structures: in addition, T has no leaves.

Theorem (Smith 1981/ Melnikov and N., 2022 in l.c. context)

If a profinite group has a co-c.e. copy as a topological structure, then it has a co-c.e. presentation as defined above.
Same for computable. The transformations are uniform.

Computably f.g. subgroups of profinite groups

- A group L is called residually finite if each $w \in L - \{e\}$ remains $\neq e$ in some finite quotient of L.
- A f.g. group L is residually finite $\iff$ it is isomorphic to a subgroup of a profinite group.

  For "$\Rightarrow$", use the profinite completion.

Theorem

A f.g. group L is isomorphic to a subgroup of some computable profinite group that is generated by finitely many computable paths $\iff$ the following two conditions hold:

- L has a Π^0_1 word problem
- L is effectively residually finite.

III. Fractal dimensions of closed subgroups

Joint with Elvira Mayordomo (arXiv:2502.09995, 2025)

Closed subgroups of a profinite group

- Write $H \leq_c G$ to express that H is a closed subgroup of $G = \varprojlim_n (G_n, p_n)$.
- Let H_n be the range of the natural projection of H into G_n.
- Let $q_n = p_n \restriction_{H_{n+1}}$. Then $H = \varprojlim_n (H_n, q_n)$, with onto maps.

Recall $G = [T]$ for the tree T associated with the inverse system. Then $H = [S]$ for its subtree S associated with (H_n, q_n).

H is a nullset for the uniform measure on $[T]$ (Haar measure), unless H has finite index (and hence is open in case that G is topologically f.g., by Nikolov and Segal 2008).

So, how can one measure the size of H? Using fractal dimensions!

Metrics on a profinite group

- Fractal dimensions are defined for bounded subsets of metric spaces. So we need a metric on G.

- The tree T for G provides us with an ultrametric:
  For distinct $g, h \in [T]$, let
  $$d(g,h) = \max\{|T_n|^{-1} \colon \; g(n) \neq h(n)\},$$
  where T_n is the n-th level of T (starting from 0).

- Problem: the tree is based on the inverse system for G, which in the general case is somewhat arbitrary.

- However, for some classes, there is a natural inverse system.

Natural metric on topologically f.g. pro-p group

A profinite group is called pro-p group if the size of every continuous finite quotient is a power of p.
If G is pro-p and (topologically) finitely generated, there is a natural inverse system:

- Let R_n be the subgroup of G generated by the p^n-th powers. Clearly $\bigcap_n R_n = \{e\}$.
- R_n is normal and open; let $G_n = G/R_n$.
- Then $(G_n, p_n)_{n\in\mathbb{N}}$, with the canonical maps $p_n\colon G_{n+1} \to G_n$, forms an inverse system for G.
- Inverse system is invariant under $\mathrm{Aut}(G)$.

Lower and upper box (or counting) dimension

Let M be a metric space, and $X \subseteq M$ be compact. For $\alpha > 0$, let $N_\alpha(X) =$ least size of a covering of X with sets of diameter $\leq \alpha$.
The lower box dimension is

$$\underline{\dim}_{\mathrm{Box}}(X) = \liminf_{\alpha \to 0^+} \frac{\log N_\alpha(X)}{\log(1/\alpha)}$$

$$\underline{\dim}_{\mathrm{Box}}(\text{Coastline}) = 1.25$$

Source: wikipedia

The upper box dimension $\overline{\dim}_{\mathrm{Box}}(X)$ is defined as the limsup.

Lower box dimension of $[S]$

Consider the metric space $[T]$ for a finitely branching tree $T \subseteq \mathbb{N}^*$.
Let $X = [S]$ where S is a subtree. (Trees will have no leaves.)

$\{[\sigma] \colon \sigma \in S_n\}$ is the "optimal covering" of $[S]$ for diameter $|T_n|^{-1}$.
Only α's of form $|T_n|^{-1}$ matter, so

$$\underline{\dim}_{\text{Box}}([S]) = \liminf_{\alpha \to 0^+} \frac{\log N_\alpha(X)}{\log(1/\alpha)} = \liminf_{n \to \infty} \frac{\log |S_n|}{\log |T_n|}$$

Example (similar to the Cantor "no middle-third" set)

- Let $T = \{0, 1, 2\}^{<\omega}$ and S the subtree of strings without a 1.
- $\log |S_n| / \log |T_n| = \log 2 / \log 3$ for each n.
- So $\underline{\dim}_{\text{Box}}[S] = \overline{\dim}_{\text{Box}}[S] = \log_3(2) \approx 0.631$

Box dimensions of closed subgroups of G

Recall that $G = [T]$. The subtree S describing $H \leq_c G$ satisfies $S_n = H_n$, where H_n is the projection of H into G_n. So

$$\underline{\dim}_{\mathrm{Box}}(H) = \liminf_{n\to\infty} \frac{\log |H_n|}{\log |G_n|}$$

Example (Barnea-Shalev 1997, essentially)

Let G be the Cantor space $\mathcal{P}(\mathbb{N})$ with symmetric difference Δ as the group multiplication. Let $G_n = \mathcal{P}(n)$.

For each $0 \leq \alpha \leq \beta \leq 1$ there is a closed subgroup H with

$$\underline{\dim}_{\mathrm{Box}}(H) = \alpha \text{ and } \overline{\dim}_{\mathrm{Box}}(H) = \beta.$$

To see this, let $R \subseteq \mathbb{N}$ be a set with lower [upper] density α [β]. Let H be the subgroup $\mathcal{P}(R)$. We have $|S_n| = 2^{|R\cap n|}$, $|T_n| = 2^n$.

Hausdorff and packing dimension

- $\dim_{\text{Hausdorff}}(X)$ is the sup of the r such that the r-dimensional Hausdorff measure $\mathcal{H}^r(X)$ is positive.

- Packing dimension $\dim_{\text{Packing}}(X)$ is defined in a similar way but "from the inside", via disjoint sets of balls with centre in X.

- We always have

$$\dim_{\text{Packing}}(X) \leq \overline{\dim}_{\text{Box}}(X)$$

$$\dim_{\text{Hausdorff}}(X) \leq \underline{\dim}_{\text{Box}}(X)$$

and also upward inequalities.

Coincidences of fractal dimensions for $[S]$

Theorem (Mayordomo and N., 2024)

Suppose a subtree S of T is levelwise uniformly branching. Then

$$\begin{aligned} \text{Packing dimension of } [S] &= \text{upper box dim. of } [S] \\ \text{Hausdorff dimension of } [S] &= \text{lower box dim. of } [S] \end{aligned}$$

- The proof uses two versions of the point-to-set principle in general metric spaces (J. Lutz, N. Lutz and Mayordomo, 2023). We will discuss this proof on the next two slides.

- For both equalities, we also have direct proofs of the inequalities $\geq$; the inequalities $\leq$ always hold.

- E.g., Tricot: $\dim_P[S] \geq \inf\{\overline{\dim}_B(V) : V \neq \emptyset \text{ open in } [S]\}$.

Proof of Theorem: constructive dimensions

Consider a computable metric space M with a dense sequence of designated points, encoded by binary strings.

The lower constructive dimension of a **point** $x \in M$ is defined by

$$cdim(x) := \liminf_{\alpha \to 0^+} C_\alpha(x) / \log(1/\alpha),$$

where $C_\alpha(x)$ is the least complexity of a designated point within radius α. The upper dimension $cDim(x)$ is defined using the sup.

Proposition (Mayordomo and N)

Let T be a computable tree. Let S be a computable subtree of T.

- $cdim(f) \le \underline{\dim}_{\text{Box}}(S)$ and $cDim(f) \le \overline{\dim}_{\text{Box}}(S)$ for each $f \in [S]$.
- If S is uniformly branching, then equalities hold when f is Martin-Löf random in $[S]$ w.r.t. the uniform measure on $[S]$.

Proof of Theorem: Point-to-set principles

Two point-to-set principles of Lutz, Lutz and Mayordomo (2023) determine the Hausdorff and packing dimension of a set $X \subseteq M$ in terms of the relativised algorithmic dimensions of the points in it:

$$\begin{aligned}
\dim_{\text{Hausdorff}}(X) &= \min_{A} \sup_{x \in X} \mathit{dim}^{A}(x), \\
\dim_{\text{Packing}}(X) &= \min_{A} \sup_{x \in X} \mathit{Dim}^{A}(x)
\end{aligned}$$

Using the previous proposition in relativised form, this shows

$$\dim_{\text{Hausdorff}}([S]) = \underline{\dim}_{\text{Box}}([S]) \text{ and } \dim_{\text{Packing}}([S]) = \overline{\dim}_{\text{Box}}([S]),$$

whenever S is a levelwise uniformly branching subtree of T.

Apply this to profinite groups G: purely geometric proof of a result that describes the Hausdorff dimension of closed subgroups of G.

Theorem (Barnea-Shalev, 1997)

Let $G = \varprojlim_n G_n$. Suppose that $H \leq_c G$.
Let H_n be the projection of H into G_n. Then

$$\dim_{\text{Hausdorff}}(H) = \underline{\dim}_{\text{Box}}(H) = \liminf_{n\to\infty} \frac{\log|H_n|}{\log|G_n|}$$

- They used Prop 2.6 in the topological algebra paper "Subgroups and subrings of profinite rings" by Abercrombie (1994).
- Our argument shows that this only needs to use the tree structures.

By our methods, we also obtain

$$\dim_{\text{Packing}}(H) = \overline{\dim}_{\text{Box}}(H) = \limsup_{n\to\infty} \frac{\log|H_n|}{\log|G_n|}.$$

Dimension spectrum of a f.g. pro-p group

- Important topic of the Barnea-Shalev 1997 paper and its sequels (such as B. Klopsch and co-authors) is the spectrum, namely, the set of possible dimensions of closed subgroups.
- For instance, the spectrum of $\mathbb{Z}_p^2$ is $\{0, \frac{1}{2}, 1\}$.
- For especially nice pro-p-groups known as p-adic analytic, the Hausdorff dimension of a closed subgroup is k/n, where k is its dimension as a manifold over $\mathbb{Z}_p$, and the whole group as a manifold has dimension n.
- There is a f.g. pro-p group with "normal" spectrum $[0, 1]$ (de las Heras and Klopsch, 2022).
- Open question: among the f.g. pro-p groups, are the p-adic analytic ones the only ones with finite spectrum?

IV. Algorithmic randomness in computable profinite groups

Joint with Willem Fouché and Matteo Vannacci

Haar measure

Any compact separable group has a unique translation invariant probability measure, called its Haar measure, we denote by μ.

If $G = \varprojlim_n (G_n, p_n)$ is profinite, this is the uniform measure on $[T]$, where T is the tree given by the inverse system.

If G is computable and infinite, the usual algorithmic test notions for Cantor space can be extended to the space of paths $[T]$.

- Kurtz random: in *no* effectively closed null set.
- Schnorr random: in *no* null set of the form $\bigcap_m G_m$, where $(G_m)_{m\in\mathbb{N}}$ is a sequence of uniformly Σ^0_1 sets, and μG_m is a uniformly computable real.

“Almost everywhere” results for k-tuples

An “almost everywhere” result for a profinite group G asserts that μ^k-almost every k-tuple $\overline{g}$ satisfies a given property.

- For $\overline{g} \in G^k$, by $\langle \overline{g} \rangle$ one denotes the closure of the subgroup of G generated by $\overline{g}$.
- Let $G = \widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n!\mathbb{Z}$ be the free profinite group of rank 1.

Some “ almost everywhere” results for $\widehat{\mathbb{Z}}$ (Jarden, Lubotzky):

(1) $\langle g \rangle$ has infinite index in $\widehat{\mathbb{Z}}$ for a.e. $g \in \widehat{\mathbb{Z}}$.

(2) $\langle \overline{g} \rangle$ has finite index in $\widehat{\mathbb{Z}}$ for for a.e. $\overline{g} \in (\widehat{\mathbb{Z}})^k$, where $k \geq 2$.

Algorithmic version

Recall " almost everywhere" results for $\widehat{\mathbb{Z}}$ (Jarden, Lubotzky):
(1) $|\widehat{\mathbb{Z}} : \langle g \rangle| = \infty$ for a.e. $g \in \widehat{\mathbb{Z}}$.
(2) $|\widehat{\mathbb{Z}} : \langle \overline{g} \rangle| < \infty$ for a.e. $\overline{g} \in (\widehat{\mathbb{Z}})^k$, where $k \geq 2$.

Theorem (Algorithmic versions of these results)

(1) If $g \in \widehat{\mathbb{Z}}$ is Kurtz random, then $|\widehat{\mathbb{Z}} : \langle g \rangle| = \infty$

(2) If $k \geq 2$ and $\overline{g} \in \widehat{\mathbb{Z}}^k$ is Schnorr random, then $|\widehat{\mathbb{Z}} : \langle \overline{g} \rangle| < \infty$; Kurtz randomness is not sufficient.

When a k-tuple generates an open subgroup a.s.

We say that a profinite group G is a k-group if $|G\colon \langle \overline{g} \rangle| < \infty$ almost surely for $\overline{g} \in G^k$.
This means $Q(G, k) = 1$ in the notation of A. Mann (1996).

- Each k-group is topologically finitely generated.
- By the results above, $\widehat{\mathbb{Z}}$ is a 2-group, and not a 1-group.
- Nikolov and Segal (2008): each subgroup of finite index in a topologically f.g. profinite group is open.
- So in the definition above one could require as well that $\langle \overline{g} \rangle$ be open.

When a k-tuple generates an open subgroup a.s.

Recall a profinite G is a k-group if $|G \colon \langle \overline{g} \rangle| < \infty$ almost surely for $\overline{g} \in G^k$.

Proposition

Let the computable profinite group G be a k-group.
Then $|G : \langle \overline{g} \rangle| < \infty$ for each weakly 2-random $\overline{g} \in G^k$.

Proof:

- Let $V_m = \{\overline{g} \in G^k \colon |G : \langle \overline{g} \rangle| \geq m\}$.
- If $\overline{g} \in V_m$ this becomes apparent at some G_n in the inverse system. So V_m is uniformly Σ^0_1.
- Also $\mu^k(V_m) \to_m 0$ since G is a k-group.
- So $(V_m)_{m \in \mathbb{N}}$ is a weak 2-test. $\square$

Work in progress with Vannacci would show that if G is pro-p, then Schnorr randomness suffices.

Effective form of a.e. results for $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

We give algorithmic versions of "a.e." theorems from
Fried and Jarden, Field arithmetic (3d edition, 2005).

- $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) = \mathrm{Aut}(\overline{\mathbb{Q}}, +, \times)$ is the absolute Galois group of $\mathbb{Q}$. A Galois group is always profinite.
- Since $\mathbb{Q}[X]$ has a splitting algorithm, G is computable.

Theorem (algorithmic form of Thm. 18.5.6 in Fried-Jarden)

Let $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let $\overline{g} \in G^k$ be Kurtz random.
Then $\langle \overline{g} \rangle$ is a free profinite group of rank k.

Effective form of a.e. results for $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group of $\mathbb{Q}$.

A field L is pseudo-algebraically closed (PAC) $\iff$ every absolutely irreducible polynomial $p \in L[X, Y]$ has a zero in L.

Theorem (algorithmic form of Thm. 27.4.8 in Fried-Jarden)

Let $g \in G$ be Kurtz random. Then the fixed field of the least closed normal subgroup containing g is PAC.

- Since Kurtz randomness is enough, these Fried-Jarden results prove more than what they say.
- For instance, weakly 1-generic also suffices.

Summary

Algorithmic presentations of profinite groups

Reviwed co-c.e. and computable profinite groups. Characterized f.g. subgroups of computable profinite groups by Π^0_1 word problem and e.r.f.

Fractal dimensions of closed subgroups

Showed coincidence of Hausdorff and lower box dimension of $[S]$ when S is a uniformly branching subtree of T. Applied it for a geometric proof of Barnea-Shalev formula for Hausdorff dimension of closed subgroups. (arXiv: 2502.09995)

Algorithmic randomness in computable profinite groups

Algorithmic versions of a.e. theorems. Often the weak notion of Kurtz randomness suffices.