Oligomorphic groups and first-order interpretations between omega-categorical structures

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What are omega-categorical structures?

Definition

Let M be an infinite countable structure M in a countable signature. M is called omega-categorical if its first-order theory determines it up to isomorphism among the countable structures.

Examples of ω-categorical structures

$(\mathbb{Q}, <)$, "the" countable dense linear order without endpoints

The Rado graph $R = R_2$. More generally, R_n, the random graph with n edge colors. Even more generally, relational Fraïsse limits.

What are omega-categorical theories?

Definition

Let T be a first-order theory without finite models in a countable signature. T is called omega-categorical if all its countable models are isomorphic.

Theorem (Ryll-Nardzewski, Svenonius, Engeler)

T is ω-categorical $\iff$ for each n, the Boolean algebra of formulas in n variables, modulo equivalence under T, is finite.

What are oligomorphic groups?

Definition

$\mathrm{Sym}(\mathbb{N})$ denotes the group of permutations of $\mathbb{N}$, with the topology of pointwise convergence:

$\lim_{r \in \mathbb{N}} \pi_r = \pi$ if for each $m \in \mathbb{N}$, $\pi_r(m)$ eventually equals $\pi(m)$.

Definition (Peter Cameron, 1980s, Book in 1990)

A closed subgroup G of $\mathrm{Sym}(\mathbb{N})$ is called oligomorphic if for each k, its componentwise action on $\mathbb{N}^k$ has only finitely many orbits.

Basic fact linking permutation groups and model theory

A closed subgroup G of $\mathrm{Sym}(\mathbb{N})$ is oligomorphic $\Longleftrightarrow$

$G = \mathrm{Aut}(M)$ for some ω-categorical structure with domain $\mathbb{N}$.

Basic fact, recall

A closed subgroup G of $\mathrm{Sym}(\mathbb{N})$ is oligomorphic $\Longleftrightarrow$

$G = \mathrm{Aut}(M)$ for some ω-categorical structure with domain $\mathbb{N}$.

Proof.

$\Rightarrow$: M is the orbit structure of the action of G on $\mathbb{N}^k$

$\Leftarrow$: The orbits of G on $\mathbb{N}^n$ are defined by the formulas that are atoms in the n-th LB algebra of $\mathtt{Th}(M)$. Only finitely many atoms.

$\square$

Oligomorphic groups versus t.d.l.c. groups

Oligomorphic groups need to be "big", to make the number of k-orbits finite for each k. Intuitively, this means an ω-categorical structure M has a lot of symmetry.

Totally disconnected, locally compact groups

- Locally compact closed subgroups of $\mathrm{Sym}(\mathbb{N})$ (called t.d.l.c.) form another much studied class:
  van Dantzig (1931); Willis (1995), ...
- They are the permutation groups with "finite sub-orbits". That is, for some finite set $S \subseteq \mathbb{N}$, each 1-orbit of the pointwise stabiliser of S is finite.
- Example: $\mathrm{Aut}(\Gamma)$ for a locally finite graph Γ. In this case one can let S be any singleton.

Being oligomorphic and being t.d.l.c. are incompatible.

What's this all about?

In our context of ω-categoricity, we want to study complex mathematical objects syntactically via FO.

Objects: Continuous maps between automorphism groups G of ω-categorical structures; the topological group $\mathrm{Aut}(G)$.

Syntax: first-order interpretations, bi-interpretations.

We can get topology on the space of interpretations from a tree representation.

Why do we want to study such objects?

- Kechris, N., Tent, JSL, 2018 started a *programme*:
  given a Borel class $\mathcal{C}$ of closed subgroups of $\mathrm{Sym}(\mathbb{N})$, determine the complexity (in the sense of Borel reducibility) of topological isomorphism between groups in $\mathcal{C}$.
- A main problem is to do this for $\mathcal{C} =$ oligomorphic groups.
- N., Schlicht, Tent, JML, 2021: upper bound E_∞
- but it could have real number invariants ("smooth")

We will see that topological isomorphism of oligomorphic groups is equivalent to bi-interpretability of the underlying ω-categorical structures. So the syntactic approach can help.

The plan

Structures and theories will be ω-categorical.

Dramatis personæ: a structure M, its first-order theory T, and its automorphism group G (a topological group).

Plan for the rest of the talk:

1. Interpreting one structure in another
2. Bi-interpretability of structures is equivalent to topological isomorphism of their automorphism groups
3. N_G/G is profinite, where $G = \mathrm{Aut}(M)$. $\mathrm{Out}(G)$ is totally disconnected locally compact.
4. $\mathrm{BI}(T) \cong \mathrm{Out}(G)$ for any theory T, where $M \models T$, and $BI(T)$ is the group of self-bi-interpretations of T up to homotopy

1. Interpreting a structure in another

(a main concept of model theory).

Interpretations

In the next three slides, consider any structures. An interpretation Γ of a structure A in a structure B is given by a dimension $k \in \mathbb{N}$ and formulas in the language of B

- $\phi_D(\overline{x})$ with k free variables
- $\phi_E(\overline{x}_1, \overline{x}_2)$ with $2k$ free variables
- $\phi_R(\overline{y}_1, \ldots, \overline{y}_n)$ for each n-ary relation symbol in the language of A, with nk free variables
- ϕ_f for each function symbol f in the language of A with $n(k+1)$ free variables

such that $A \cong \Gamma(B)$, where $\Gamma(B)$ is the structure declared on $\phi_D(B)/\phi_E(B)$ using the ϕ_R and ϕ_f.

Interpretations: quotient field

Let R be a domain. The construction of its quotient field K can be seen as an interpretation Γ so that $K = \Gamma(R)$. Everything below must be first-order definable in the language of rings.

- The domain D is the binary relation $\{(a, b) \colon b \neq 0\}$ ("fractions")
- The equivalence relation E holds of pairs $(a, b), (c, d)$ in D if $ad = bc$ ("when do two fractions express the same?")
- $(a, b) + (c, d) = (ad + bc, bd)$ well-defines addition on D/E
- $(a, b) \times (c, d) = (ac, bd)$

Bi-interpretability of structures

Definition

Structures A, B are bi-interpretable (BI) if there are first-order interpretations Γ, Δ such that

- $A \cong \Gamma(B)$ and $B \cong \Delta(A)$
- some isomorphism $\gamma : A \cong \widehat{A} = \Gamma(\Delta(A))$ is definable in A; some isomorphism $\delta \colon B \cong \Delta(\Gamma(B))$ is definable in B

Here $\widehat{A}$ consists of equivalence classes of $n \times k$-tuples from A, and γ is given as an $1 + n \times k$-ary definable relation in A.

Rings of integers and rationals are bi-interpretable

Continue the previous example: suppose that domain R is definable in K (such as $\mathbb{Z}$ in $\mathbb{Q}$). Then R and K are bi-interpretable.

Proof: Let $\Delta(K)$ be the copy of R given by its f.o. definition in K, with the operations of K restricted to it.

The isomorphism $\gamma : K \cong \widehat{K} = \Gamma(\Delta(K))$ is definable in K.

The isomorphism $\delta \colon R \cong \widehat{R} = \Delta(\Gamma(R))$ is definable in R.

2. BI of omega-categorical structures

is equivalent to

topological isomorphism of

their automorphism groups

Theorem (Coquand unpubl.; Ahlbrandt and Ziegler 1986)

For ω-categorical structures A, B

$$A \text{ and } B \text{ are bi-interpretable} \iff \mathrm{Aut}(A) \cong \mathrm{Aut}(B) \ .$$

Example

Let $A = (0,1)_{\mathbb{Q}}$, $B = [0,1]_{\mathbb{Q}}$ as linear orders.

Clearly $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$, so A and B are bi-interpretable.

Direct proof: Interpretation of A in B is obvious.
For interpretation of B in A, use domain $D = A \times A$, and choose ϕ_E and $\phi_<$ so that

- all pairs $a < b$ denote the left endpoint,
- all pairs $a > b$ denote the right endpoint.
- pairs (c, c) denote the other elements.

Variants of Coquand's result

- End(A) : the topological monoid of endomorphisms of A.
- Pol(A) : the polymorphism clone of A. This is a many-sorted topological structure consisting of all the homomorphisms $A^k \to A$, for $k \geq 1$, one sort per k, and compositions.

A formula is existential positive if it has no $\forall$, no $\neg$; primitive positive if of the form $\exists \dots \exists$(conjunction of atomic)

Theorem (Bodirsky and Junker 11; Bodirsky and Pinsker 15)

For ω-categorical structures A, B,

- A and B are existential positive bi-interpretable $\iff$ $\mathrm{End}(A) \cong \mathrm{End}(B)$ (assuming no constant endomorphisms).
- A and B are positive primitive bi-interpretable $\iff$ $\mathrm{Pol}(A) \cong \mathrm{Pol}(B)$.

When bi-interpretability fails

1. $(\mathbb{Q}, <)$ and the random graph R_2 are not bi-interpretable:

$\mathrm{Aut}(\mathbb{Q}, <)$ has three nontrivial normal subgroups, while $\mathrm{Aut}(R_2)$ only has trivial normal subgroups (Truss, 1985).

So these automorphism groups aren't even algebraically isomorphic.

2. (Khoussainov) Let $L = [0, 1)_{\mathbb{Q}}$. Define a p.o. $\preceq$ on $L \times L$ by

$\langle x_0, y_0 \rangle \preceq \langle x_1, y_1 \rangle$ if $y_0 = 0 \wedge x_0 \leq x_1$, or $x_0 = x_1 \wedge y_0 \leq y_1$.

- $(L, \leq)$ is interpretable in $(L \times L, \preceq)$ and conversely.
- But the two structures have nonisomorphic automorphism groups, so they are not bi-interpretable.

3. Two topological groups associated with G

N. and Paolini, 2024, arxiv.org/pdf/2410.02248

Profinite groups from oligomorphic groups

The normaliser of G in $\mathrm{Sym}(\mathbb{N})$ is

$$N_G = \{\sigma \in \mathrm{Sym}(\mathbb{N})\colon \sigma^{-1} G \sigma = G\}.$$

It coincides with $\mathrm{Aut}(\mathcal{E}_M)$ where $\mathcal{E}_M$ is the orbital structure of M. This $\mathcal{E}_M$ has a 2k-ary eqrel for each $k \geq 1$, saying that the two tuples are in the same k-orbit of G.

Theorem (N. and Paolini, 2024)

N_G/G is compact and totally disconnected (i.e., profinite).

Any separable profinite group occurs (Evans and Hewitt, 1991).

Out(G) is totally disconnected., locally compact

$\mathrm{Aut}(G)$ is the group of continuous automorphisms of G.
It carries a unique Polish topology that makes its action on G continuous (N. and Schlicht).

Theorem (N. and Paolini, 2024)

1. The group of inner automorphisms $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$.
2. The group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ with the quotient topology is totally disconnected, locally compact (t.d.l.c.).
3. The image of the natural map $N_G \to \mathrm{Aut}(G)$ induces a compact open subgroup as predicted by van Dantzig.
4. If M has no algebraicity then this group equals $\mathrm{Out}(G)$. So $\mathrm{Out}(G) \cong N_G/GC_G$.

Examples for (4): $(\mathbb{Q}, <)$, random graph.

4. The group of bi-interpretations of T

modulo homotopy

is isomorphic to Out(G)

Joint with P. Schlicht, to appear on arXiv

Interpretation of a theory U in a theory T (a)

Let T, U be relational theories. An interpretation $\alpha \colon T \rightsquigarrow U$ is given by objects (a) and (b):

(a.i) a dimension $k \geq 1$ [Notation: for each variable z we have a k-tuple of variables $\overline{z} = (z_1, \ldots, z_k)$]

(a.ii) formulas $\phi_D(\overline{x})$ and $\phi_E(\overline{x}, \overline{y})$ in the language of T such that

$$T \vdash \text{“}D \neq \emptyset\text{”}, \text{ and } T \vdash \text{“}E \text{ is an equivalence relation on } D\text{”}.$$

(a.iii) For each atomic formula $R(x_1, \ldots, x_n)$ in the language of U, excluding equality, a formula $\alpha.R\,(\overline{x}_1, \ldots, \overline{x}_n)$ in the language of T such that T proves its invariance under the equivalence relation E.

Interpretation of a theory U in a theory T (b)

For any formula ϕ in the language of U, we define $\alpha.\phi$ to be the formula in the language of T obtained as follows.

(b.i) replace each atomic formula $R(y_1, \ldots, y_n)$ with $\alpha.R(\overline{y}_1, \ldots, \overline{y}_n)$,

(b.ii) replace $x = y$ with $\phi_E(\overline{x}, \overline{y})$,

(b.iii) replace each quantifier $\forall x$ (or $\exists x$) with k quantifiers of the same type over the components of $\overline{x}$.

We require the following condition to hold:

$$C_{T,U}(\alpha): \qquad \text{for every sentence } \psi \in U, \text{ we have } \alpha.\psi \in T.$$

Interpretations of U in T as a path space of a tree

- The set of interpretations $T \rightsquigarrow U$ can be seen as the set of paths of a subtree $B_{T,U}$ of $\omega^{<\omega}$, as follows.
- Only the root is infinitely branching. The first branching encodes the triples $c = \langle k, \phi_D, \phi_E \rangle$ that T allows.
- Each further level (the set of strings of a length $k > 1$) is dedicated to a relation symbol R of L_U: the extension on that level decides which formula of L_T the symbol $Rx_1 \dots x_n$ is mapped to (the formula needs to be invariant under E according to T).
- Let $B_{T,U}$ be the tree of strings σ such that the condition $C_{T,U}(\sigma)$ holds for each sentence ψ such that all relation symbols of U occuring in it have been assigned by σ.
- The paths space $[B_{T,U}]$ is locally compact.

Picture of the tree B(T,U)

A closer look at Ahlbrandt and Ziegler

Theorem (Recall)

A and B are bi-interpretable $\iff \mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

Ahlbrandt and Ziegler for their proof used a category of ω-cat structures. An interpretation of B in A is now a morphism $\gamma\colon A \rightsquigarrow B$ decoding B from A.

They viewed Aut as a functor mapping γ to a continuous homomorphism $F = \mathrm{Aut}(\gamma)\colon \mathrm{Aut}(A) \to \mathrm{Aut}(B)$.

- The continuous F one gets are precisely the ones such that the action of A on B via F has only finitely many 1-orbits.
- $\mathrm{Aut}(\gamma) = \mathrm{Aut}(\delta)$ iff there is an A-definable bijection η between the domains such that $\delta = \gamma \circ \eta$. (homotopic)

Picture to explain this

Let $\gamma, \delta \colon A \rightsquigarrow B$. Then $\mathrm{Aut}(\gamma) = \mathrm{Aut}(\delta)$ iff there is an A-definable bijection η between the domains such that $\delta = \gamma \circ \eta$.

Homotopy of interpretations of theories

Given interpretations $\alpha_0, \alpha_1 \colon T \rightsquigarrow U$ where α_i is based on (formulas defining) sorts D_i/E_i in any model.
Write $\alpha_0 \approx_T \alpha_1$ if for some formula ϕ_Θ, the theory T contains the sentence expressing that

$$\phi_\Theta \text{ defines a bijection } \Theta \colon D_0/E_0 \to D_1/E_1,$$

and for each n-ary relation symbol R of L_U, the theory T contains the sentence expressing that

$$\Theta^n(\alpha_0.R) = \alpha_1.R.$$

Theories with composition of interpretations form a category. We obtain a quotient category taking morphisms up to homotopy.

Topologise the quotient category

- Recall $[B_{T,U}]$ is a t.d.l.c. space.
- The projection $q\colon [B_{T,U}] \to \mathrm{Mor}(T,U)$ that sends α to its equivalence class $[\alpha]_{\approx}$ induces quotient topology.
- We prove that q is open.
- So $\mathrm{Mor}(T,T)$ is a locally compact monoid. Also totally disconnected.

The group of invertible elements of $\mathrm{Mor}(T,T)$ is denoted $\mathrm{BI}(T)$. This can be seen as a closed set in $\mathrm{Mor}(T,T) \times \mathrm{Mor}(T,T)$, and thus is a t.d.l.c. group.

Describe Out(G) using FO

$\mathrm{BI}(T)$ is the group of self bi-interpretations of T, modulo homotopy. It is t.d.l.c.

Theorem (N. and Schlicht, 2026)

$\mathrm{Out}(G)$ is topologically isomorphic to $\mathrm{BI}(T)$, and hence t.d.l.c.

Corollary

If the signature of M is finite then the group $\mathrm{Out}(G)$ is discrete.

Out(G) for some structures without algebraicity

By M. Rubin, each BI is of dimension 1, up to homotopy.

$M = (\mathbb{Q}, <)$: $\mathrm{Out}(G)$ has one non-trivial element. It is given as the equivalence class of the permutation in N_G that sends x to $-x$. The corresponding bi-interpretation has dimension 1, and switches the 2-orbits $\{\langle a, b\rangle : a < b\}$ and $\{\langle a, b\rangle : a > b\}$

$M = R$: $\mathrm{Out}(G)$ has one non-trivial element. It is given as the equivalence class of the permutation in N_G that is an isomorphism of R and the random graph where the edges are the non-edges of R. The corresponding bi-interpretation has dimension 1, and switches the 2-orbits $\{\langle a, b\rangle : aRb\}$ and $\{\langle a, b\rangle : \neg aRb\}$.

Open questions

- Is BI on omega-categorical structures smooth?
  That is, can we assign complete real invariants to the oligomorphic groups?

- Can $\mathrm{Out}(G)$ fail to be compact?
  That is, is there T sych that there are self bi-interpretations of T of arbitrarily high dimension, up to homotopy?