20 years of K-triviality

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2002-2012

- In June 2012, I gave a talk at Chicheley Hall entitled "10 Years of Triviality", in connection with the Turing year.
- By "triviality" I meant K-triviality, a property of sets of natural numbers introduced in 1975 by Chaitin and Solovay.
- Intuitively, this property says that the set is

  "far from random",

  in the sense that the prefix free descriptive complexity of the initial segments grows as slowly as possible.

2012-2022

This talk provides background, and then traces the developments on K-trivial sets from 2012 to the present.

Describe Five further characterisations of K-triviality.

Cover In particular, the covering problem was solved in the affirmative: for c.e. sets, K-trivial is the same as being below an incomplete ML-random.

Ideals A dense hierarchy of Turing ideals in the K-trivials was found and its relationship to cost functions studied.

$\leq_{ML}$ ML-reducibility promises a better understanding of the internal structure of the K-trivials.
There is an ML-complete K-trivial.

Measures K-trivial and C-trivial measures.

Descriptive string complexity K

A partial computable function from binary strings to binary strings, called machine, is prefix-free if its domain is an antichain under the prefix relation of strings.

There is a universal prefix-free machine $\mathbb{U}$: for every prefix-free machine M,

$$M(\sigma) = y \text{ implies } \mathbb{U}(\tau) = y \text{ for some } \tau \text{ with } |\tau| \leq |\sigma| + d_M,$$

and the constant d_M only depends on M.

The prefix-free Kolmogorov complexity of a string y is the length of a shortest $\mathbb{U}$-description of y:

$$K(y) = \min\{|\sigma| : \mathbb{U}(\sigma) = y\}.$$

Martin-Löf randomness (1966)

Sets are viewed as points in Cantor space $\{0,1\}^{\mathbb{N}}$.
Let λ denote the uniform (product) measure on $\{0,1\}^{\mathbb{N}}$.

- A ML-test is a uniformly Σ^0_1 sequence $(G_m)_{m\in\mathbb{N}}$ of open sets in $\{0,1\}^{\mathbb{N}}$ such that $\lambda G_m \le 2^{-m}$ for each m.
- A set Z is ML-random if Z passes each ML-test, in the sense that Z is not in all of the G_m.

There is a universal ML-test (S_r): a set Z is ML-random iff it passes (S_r). MLR denotes the class of ML-random sets.

Weak 2-randomness is defined by passing tests of a more general kind: replace the condition $\lambda G_m \le 2^{-m}$ by $\lim_m \lambda G_m = 0$.

K-trivials (1975)

The Schnorr-Levin theorem states that

$Z \in 2^{\mathbb{N}}$ is ML-random if and only if $K(Z \upharpoonright n) \ge^+ n$.

In the other extreme:

Definition (Chaitin, 1975)

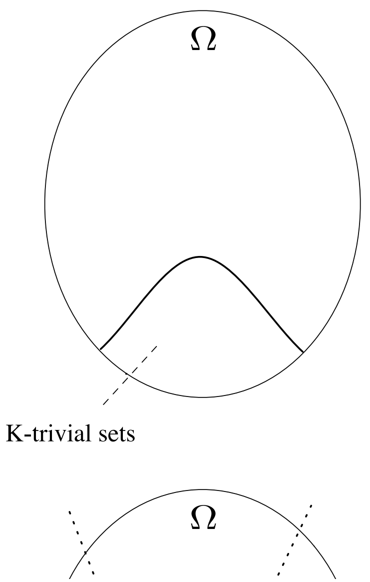
$A \in 2^{\mathbb{N}}$ is K-trivial if $K(A \upharpoonright n) \le^+ K(n)$ for each n.

- computable $\Rightarrow$ K-trivial $\Rightarrow$ Δ^0_2 (Chaitin)
- Solovay, '75: there is a noncomputable K-trivial set.

Downey, Hirschfeldt, N., Stephan, 2003

If A and B are K-trivial, then $A \oplus B$ is K-trivial.

Some properties of the K-trivials



Some researchers who have worked on K-triviality from 2002 on

- B Laurent Bienvenu
- D Rod Downey
- G Noam Greenberg
- H Denis Hirschfeldt
- K Antonin Kučera
- M Joseph Miller
- N André Nies
- S Frank Stephan
- T Dan Turetsky

17 characterisations of the K-trivials

The characterisations are according to four paradigms:

- highly compressible initial segments (definition and one more)
- weak as an oracle (13)
- computed by many (2)
- has computable approximation with few changes (1).

C1: Low for K, C2: low for ML-randomness

Theorem (N.-Hirschfeldt; N 03)

The following are equivalent for $A \in 2^{\mathbb{N}}$:

1. A is K-trivial.
2. $K^A =^+ K$ (A is low for K).
3. $\mathsf{MLR}^A = \mathsf{MLR}$ (A is low for ML-randomness).

$1 \Rightarrow 2$ uses the golden run method (N 03). The same method shows many of the properties of the K-trivials shown earlier.

C3: basis for ML-rd., C4: $\Delta^0_2 \cap$ low for Ω

Theorem (C3: HNS 06)

$A \in 2^{\mathbb{N}}$ is K-trivial $\iff A \leq_T Z$ for some $Z \in \mathsf{MLR}^A$.

"$\Rightarrow$" K-triviality coincides with lowness for ML-randomness. By the Kučera-Gacs Theorem each set that is low for ML-randomness is a basis for randomness.

"$\Leftarrow$" for this we introduced the "hungry sets construction".

- $\Omega = \sum\{2^{-|\sigma|} \colon \mathbb{U}(\tau)\text{ halts}\}$ is Chaitin's halting probability, which is ML-random and Turing complete.
- Call $A \subseteq \mathbb{N}$ low for Ω if Chaitin's Ω is ML-random in A.

Corollary (C4: HNS 06)

A is K-trivial $\iff A$ is Δ^0_2 and low for Ω.

C5-C7: Characterisations as classes $\mathrm{Low}(\mathcal{C}, \mathcal{D})$

For randomness notions $\mathcal{C} \subseteq \mathcal{D}$, one says that A is $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ if $\mathcal{C} \subseteq \mathcal{D}^A$. That is, A shrinks $\mathcal{D}$ by so little that is still contains $\mathcal{C}$. Let

$$\mathsf{W2R} \subseteq \mathsf{MLR} \subseteq \mathsf{CR}$$

denote the classes of weakly 2 randoms, ML-randoms, and computably randoms, respectively.

Theorem (C5-C7: N 03; N. 09; DN Weber and Yu, 06)

Let $\mathcal{C} \subseteq \mathcal{D}$ be randomness notions among $\mathsf{W2R}, \mathsf{MLR}, \mathsf{CR}$.
Each of the classes $\mathrm{Low}(\mathcal{C}, \mathcal{D})$ coincides with K-triviality, except that $\mathrm{Low}(\mathsf{CR}, \mathsf{CR}) =$ computable.

C8,C9: relativizing the difference left-c.e. reals

Recall that a real α is left-c.e. if $\alpha = \sup_s q_s$ for a computable sequence $\langle q_s \rangle_{s\in\mathbb{N}}$ of rationals.

We say that a real α is difference left-c.e. if $\alpha = \beta - \gamma$ for left-c.e. reals α, β. (These reals form a real closed field.)

Theorem (C8: DHMN 05, also see N. Book 5.5.14)

A is K-trivial $\iff$ Ω^A is left-c.e. $\iff$ each prefix free machine relative to A has a difference left-c.e. halting probability.

Corollary (C9: J. Miller)

A is K-trivial $\iff$ each real that is difference left-c.e. relative to A, is difference left-c.e.

C10: via Martin-Löf noncuppability

$A \in \Delta^0_2$ is ML-noncuppable if $A \oplus Z \geq_T \emptyset'$ implies $Z \geq_T \emptyset'$ for each ML-random Z. Otherwise, A is ML-cuppable.

Fact: If $A \in \Delta^0_2$ is not K-trivial then A is not a base for ML-randomness, so $Z := \Omega^A \not\geq_T A$, so A is ML-cuppable.

Theorem (C10: Day and Miller, 2012)

A is K-trivial $\iff$ A is ML-noncuppable

We say that a real z is a positive density point if $\underline{\rho}(E \mid z) > 0$ for every effectively closed $E \ni z$. Day and Miller used the following characterisation of the incomplete ML-random reals via density.

Theorem (BHMN, 11)

For a Martin-Löf random real z,

$$z \not\geq_T \emptyset' \iff z \text{ is a positive density point.}$$

C11: via Martin-Löf covering

Proposition (HNS 06)

If c.e. set A is below a Turing incomplete ML-random Z, then Z is ML-random in A, so A is K-trivial.

Theorem (C11: BGKNT 16 & Day, Miller 15)

Let A be a c.e. set. Then A is K-trivial $\iff$ A is computable from some Turing incomplete ML-random.

- BGKNT 16 introduced Oberwolfach (OW) randomness, a slight strengthening of ML-randomness. They showed that each ML-random, non OW-random Z computes each K-trivial.
- Day and Miller provided such a set Z which is Δ^0_2.
- So, there is a single incomplete Δ^0_2 ML-random above all the K-trivials!

Definition

A cost function is a computable function $\mathbf{c}\colon \mathbb{N}^2 \to \mathbb{R}^{\geq 0}$ satisfying:
$\mathbf{c}(x,s) \geq \mathbf{c}(x+1,s)$ and $\mathbf{c}(x,s) \leq \mathbf{c}(x,s+1)$;
$\underline{\mathbf{c}}(x) = \lim_s \mathbf{c}(x,s) < \infty$;
$\lim_x \underline{\mathbf{c}}(x) = 0$ (the limit condition).

Definition

Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A.
Let $\mathbf{c}$ be a cost function. The total cost $\mathbf{c}(\langle A_s \rangle)$ is

$$\sum_s \mathbf{c}(x,s) [\![x \text{ is least s.t. } A_s(x) \neq A_{s-1}(x)]\!].$$

A Δ^0_2 set A obeys a cost function $\mathbf{c}$ if there is some computable approximation $\langle A_s \rangle$ of A for which the total cost $\mathbf{c}(\langle A_s \rangle)$ is finite.

Write $A \models \mathbf{c}$ for this. FACT: There is a c.e., noncomputable $A \models \mathbf{c}$.

C12: Dynamic characterisation

Summarize: a Δ^0_2 set obeys $\mathbf{c}$ if it can be computably approximated obeying the "speed limit" given by $\mathbf{c}$.

Let $\mathbf{c}_\Omega(x,s) = \Omega_s - \Omega_x$ (where $\langle \Omega_s \rangle_{s \in \mathbb{N}}$ is an increasing computable approximation of Ω).

Theorem (C12: N., Calculus of cost functions, 2017)

Let $A \in \Delta^0_2$. Then A is K-trivial $\iff$ A obeys $\mathbf{c}_\Omega$.

- Older result (N 09): A is K-trivial $\iff$ A obeys the "standard cost function" $\mathbf{c}_\mathcal{K}$ where $\mathbf{c}_\mathcal{K}(x,s) = \sum_{i<x} 2^{-K_s(i)}$.
- These results directly apply the definition of K-triviality, rather than some previously known equivalent notion.

C13: Solovay functions

Recall that $A \subseteq \mathbb{N}$ is K-trivial if $K(A \upharpoonright n) \le^+ K(n)$ for each n. Can one replace the K on the right side by a computable function?

We say that a computable function f is a Solovay function if $\forall n\, K(n) \le^+ f(n)$ and $\exists^\infty n\, K(n) =^+ f(n)$.

Solovay showed their existence. E.g. let $f(\langle x, \sigma, t\rangle) = |\sigma|$ if $\mathbb{U}(\sigma) = x$ in exactly t steps, and else some coarse upper bound of $K(x)$ such as $2 \log |x|$. In fact there is a nondecreasing Solovay function.

Theorem (C13: Bienvenu and Downey, 2009)

(a) A is K-trivial $\iff$

$K(A \upharpoonright n) \le^+ f(n)$ for each Solovay function f.

(b) There is a single Solovay function f that does it.

C14, C15: Variations on low for Ω

Theorem (C14: Greenberg, Miller, Monin, and Turetsky, 2018, together with Stephan and Yu)

A is K-trivial $\iff$

for all Y such that Ω is Y-random, Ω is $Y \oplus A$-random.

The implication $\Rightarrow$ was proved by Stephan and Yu, unpublished. GMMT obtained the converse.

Theorem (C15: GMMT, 2018)

A is K-trivial $\iff$

for all Y such that Ω is Y-random, Y is LR-equivalent to $Y \oplus A$.

The implication $\Leftarrow$ is clear via $Y = \emptyset$. The hard part is to show K-trivials have this property.

C16, C17: changing the bits at the positions in A preserves randomness

Theorem (C16, C17: Kuyper and Miller, 2017)

A is K-trivial $\iff$ $Y \triangle A$ is ML-random for each ML-random Y

$\iff$ $Y \triangle A$ is weakly 2-random for each weakly 2-random Y.

In both cases, it was know that K-triviality implies the condition, because K-triviality implies lowness for the randomness notion. The surprising fact was that this seemingly weak aspect of lowness is sufficient for K-triviality.

Internal structure of the K-trivials

ML-reducibility

- It appears that Turing reducibility is too fine to understand the structure of the K-trivials.
- A coarser "reducibility" is suggested by Kucera's early results, and the solution to the covering problem from 2014.

Recall that MLR denotes the class of Martin-Löf random sets.

Definition

For K-trivial sets A, B, we write $B \geq_{ML} A$ if

$$\forall Z \in \mathsf{MLR}\ [Z \geq_T B \Rightarrow Z \geq_T A].$$

I.e., any ML-random computing B also computes A.

Each K-trivial A is ML-equivalent to a c.e. K-trivial $D \geq_T A$ (GMNT 22). So one only needs to consider the c.e. K-trivials.

Structure of the K-trivials w.r.t. $\leq_{ML}$

- The least degree consists of the computable sets. This follows from the low basis theorem with upper cone avoiding.
- There is a ML-complete K-trivial set, called a "smart" K-trivial (BGKNT '16).
- There is a dense hierarchy of principal ideals $\mathcal{B}_q$, $q \in (0,1)_{\mathbb{Q}}$. E.g., $\mathcal{B}_{0.5}$ consists of the sets that are computed by both "halves" of a ML-random Z, namely Z_{even} and Z_{odd} (GMN 19).
- Some further interesting subclasses of the K-trivials are downward closed under $\leq_{ML}$: e.g., the strongly jump traceables, which conincide with the sets below all the ω-c.a. ML-randoms (by HGN '12, along with GMNT 22).

Degree theory for $\leq_{ML}$ on the K-trivials

Recall: $B \geq_{ML} A$ if $\forall Z \in \mathsf{MLR}\ [Z \geq_T B \Rightarrow Z \geq_T A]$.

Results from GMNT 22, arxiv 1707.00258

(a) For each noncomputable c.e. K-trivial D there are c.e. $A, B \leq_T D$ such that $A \mid_{ML} B$.

(b) There are no minimal pairs.

(c) For each c.e. A there is a c.e. $B >_T A$ such that $B \equiv_{ML} A$.

(a) is based on a method of Kučera. (b) and (c) use cost functions.

Cost functions characterising ML-ideals

Definition (Recall)

Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A.
Let $\mathbf{c}$ be a cost function. The total cost $\mathbf{c}(\langle A_s \rangle)$ is
$\sum_s \mathbf{c}(x, s)[\![x \text{ is least s.t. } A_s(x) \neq A_{s-1}(x)]\!]$.
A Δ^0_2 set A obeys a cost function $\mathbf{c}$ if there is some computable approximation $\langle A_s \rangle$ of A for which the total cost $\mathbf{c}(\langle A_s \rangle)$ is finite.

Let $\mathbf{c}_{\Omega,1/2}(x, s) = (\Omega_s - \Omega_x)^{1/2}$.

Theorem (GMN 19)

The following are equivalent:

1. A is computed by both halves of a ML-random.
2. A obeys $\mathbf{c}_{\Omega,1/2}$.

Cost functions and computing from randoms

Definition

Let $\mathbf{c}$ be a cost function. Recall $\underline{\mathbf{c}}(n) = \lim_s c(n, s)$.
A $\mathbf{c}$-test is a sequence (U_n) of uniformly Σ^0_1 subsets of $\{0, 1\}^{\mathbb{N}}$ satisfying $\lambda(U_n) = O(\underline{\mathbf{c}}(n))$.

Important yet easy fact

Suppose that Z is ML-random but is captured by a $\mathbf{c}$-test.
Suppose that A obeys $\mathbf{c}$. Then $A \leq_T Z$.

Definition (ML-completeness for a cost function, GMNT 22)

Let $\mathbf{c} \geq \mathbf{c}_\Omega$ be a cost function. We say that a K-trivial A is smart for $\mathbf{c}$ if $A \models \mathbf{c}$, and $B \leq_{ML} A$ for each $B \models \mathbf{c}$.

Theorem (GMNT 22, extending BGKNT 16 result for $\mathbf{c}_\Omega$)

For each cost fcn $\mathbf{c} \geq \mathbf{c}_\Omega$ there is a c.e. set A that is smart for $\mathbf{c}$.

We may assume that $\mathbf{c}(k) \geq 2^{-k}$. Build A. There is a particular Turing functional Γ such that it suffices to show $A = \Gamma^Y \Rightarrow Y$ fails some $\mathbf{c}$-test.

- During the construction, let $\mathcal{G}_{k,s} = \{Y : \Gamma^Y_t \upharpoonright 2^{k+1} \prec A_t \text{ for some } k \leq t \leq s\}$.
- Error set $\mathcal{E}_s$ contains those Y such that Γ^Y_s is to the left of A_s.
- Ensure $\lambda \mathcal{G}_{k,s} \leq \mathbf{c}(k, s) + \lambda(\mathcal{E}_s - \mathcal{E}_k)$. If this threatens to fail put the next $x \in [2^k, 2^{k+1})$ into A. Then $\langle \mathcal{G}_k \rangle$ is the required $\mathbf{c}$-test.

ML-completeness for a cost function

Definition (recall)

Let $\mathbf{c} \geq \mathbf{c}_\Omega$ be a cost function. We say that a K-trivial A is smart for $\mathbf{c}$ if A is ML-complete among the sets that obey $\mathbf{c}$.

Theorem (GMNT 22)

For each K-trivial A there is a cost function $\mathbf{c}_A \geq \mathbf{c}_\Omega$ such that A is smart for $\mathbf{c}_A$.

This shows that there are no ML-minimal pairs:
if K-trivials A, B are noncomputable, there is a noncomputable c.e. D such that $D \models \mathbf{c}_A + \mathbf{c}_B$. Then $D \leq_{ML} A, B$.

Smartness for $\mathbf{c}_\Omega$ and half-bases

Recall:

Theorem (BGKNT 16)

Not every K-trivial is a half-base.

Proof (different from the original one).

- Ω_{even} and Ω_{odd} are low;
- If $Y \in \mathsf{MLR}$ is captured by a $\mathbf{c}_\Omega$-test, then it is superhigh.
- So a smart K-trivial is not a half-base.

□

Descriptive complexity for measures

μ will denote a probability measure on Cantor space.

- Let $C(\mu \restriction n) = \sum_{|x|=n} C(x)\mu[x]$ be the μ-average of all the $C(x)$ over all strings x of length n.
- In a similar way we define $K(\mu \restriction n)$.

E.g. $C(\lambda \restriction n \geq n - 1$, and $K(\lambda \restriction n \geq^+ n + K(n)$.

Theorem (NS 21)

Each K-trivial [C-trivial] measure is concentrated on its atoms.

Questions

- Is being a smart K-trival an arithmetical property?
  Can a smart K-trivial be cappable?
  Can it obey a cost function stronger than $\mathbf{c}_\Omega$?
- Is $\leq_{\mathrm{ML}}$ an arithmetical relation?
  Are the ML-degrees of the K-trivials dense?
- Is there an incomplete ω-c.a. ML-random above all the K-trivials?
- Is every C-trivial measure K-trivial?
  (The answer is yes in case $K(C(n) \mid n, K(n))$ is bounded.
  I.e., there are finitely many options to compute $C(n)$ from n and $K(n)$, with one successful.)

Some references

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