Duality between topological groups and approximation groupoids

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Abstract. Given a totally disconnected topological group G, we consider M(G), the set of its open cosets. On M(G) there is a natural groupoid operation: if A is a left coset of an open subgroup U and B is a right coset of U, then AB is a coset. After adjoining the empty set, M(G) with the intersection operation forms an inductive groupoid, called the approximation groupoid of G.

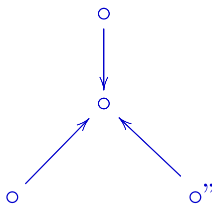
We establish duality theorems between classes of topological groups with countably many (compact) open cosets, and classes of countable approximation groupoids. We consider totally disconnected compact, locally compact, and Roelcke precompact groups. These theorems are akin to Stone duality between Stone spaces and countable Boolean algebras. All the relevant classes and maps will be Borel. In each case we reduce the continuum topological setting to a setting of countable structures.

The duality can be used to define what it means for a group in the classes above to be computable.
This builds on two previous papers with Kechris, Tent, and Schlicht.

A dichotomy in mathematics

Structures up to isomorphism

- " Let G be the graph


"

- "Let $\mathcal{H}$ be the separable infinite-dimensional Hilbert space."

Structures with presentations

- "Let G be the graph given by the list of edges
  $\{(1,0),(2,0),(3,0)\}$."

- "Let $\mathcal{H}$ be the space $\ell_2(\mathbb{C})$ of square summable sequences of complex numbers."

Complexity of membership and isomorphism

In the concrete setting, presentations can usually be seen as points in a suitable Polish space X.
Given a set $\mathcal{C} \subseteq X$, one tries to determine the complexity of:

- Membership: is a presented structure A in $\mathcal{C}$?
- Isomorphism: given $A, B \in \mathcal{C}$, is $A \cong B$?

Example

The presentations of separable C^*-algebras form a Borel subset $\mathcal{C}$ of a suitable Polish space.

Theorem (Elliott et al., Math. Res. Letters, 2013, together with Sabok, Inventiones Math., 2016)

The isomorphism relation on $\mathcal{C}$ is complete w.r.t. $\leq_B$ for orbit equivalence relation of Polish group actions on Polish spaces.

Two views of Stone duality

Abstract:

- $\mathcal{C}$ = Boolean algebras M, $\mathcal{D}$= Stone spaces G.
- The dual of M is the space of ultrafilters of M.
- The dual of G is the Boolean algebra of clopen subsets of G.

Concrete:

- Boolean algebra M has domain $\mathbb{N}$, Stone spaces are of the form $[T]$ where $T \subseteq 2^{<\omega}$ is a binary tree.
- $\mathcal{G}(M)$ = tree of strings that describe a non-0 conjunction of literals.
- $\mathcal{W}(T)$ = a list of all clopen sets in $[T]$ with no repetition, with $\cap, \cup$, complementation.

Borel duality

We have Borel classes $\mathcal{C}$ and $\mathcal{D}$ of presentations, and Borel maps

$$\begin{array}{rcl} \mathcal{W}\colon \mathcal{C} & \to & \mathcal{D} \\ \mathcal{C} & \leftarrow & \mathcal{D}\colon \mathcal{G} \end{array}$$

- E.g. in a Borel version of Stone duality, $\mathcal{C}$ consists of the Boolean algebras with domain $\mathbb{N}$, and $\mathcal{D}$ consists of the Stone spaces of the form $[T]$, where T is a binary tree.
- The maps $\mathcal{G}$ and $\mathcal{W}$ effectuating the duality are Borel.

Views of non-Archimedean groups

Abstractly,
non-Archimedean groups are separable topological groups with a ctble neighbourhood basis of 1 consisting of open subgroups.
Concretely,

- non-Archimedean groups are closed subgroups of S_∞, where S_∞ denotes the group of permutations of $\mathbb{N}$ with the topology of pointwise convergence.
- Equivalently, they are the automorphism groups of structures with domain $\mathbb{N}$.
- They form a standard Borel space, because being a subgroup is a Borel property on the Effros space of closed subsets of S_∞.

To get from abstract to concrete, let the group act from the left on the left cosets of subgroups in this neigbourhood base.

Programme (Kechris, N., Tent, JSL 2017; Logic Blog 2020; N., Schlicht, Tent JML 2021

Study natural classes $\mathcal{C}$ of closed subgroups of S_∞; in particular, study the complexity of membership and isomorphism.

Goal in this talk (and corresponding paper with Melnikov):

- contribute to this programme by providing Borel duality theorems
- they transform locally Roelcke precompact topological groups into countable structures, called approximation groupoids.
- build on work in the above-mentioned papers, but simplify setting of Nies, Schlicht and Tent 2021 by using groupoids rather than structures with a ternary relation as they did.

Definition (Roelcke, 1982)

A closed subgroup G of S_∞ is called Roelcke precompact (R.p.) if for each open subgroup U of G, there is a finite set α such that $U\alpha U = G$. I.e., U has only finitely many double cosets.

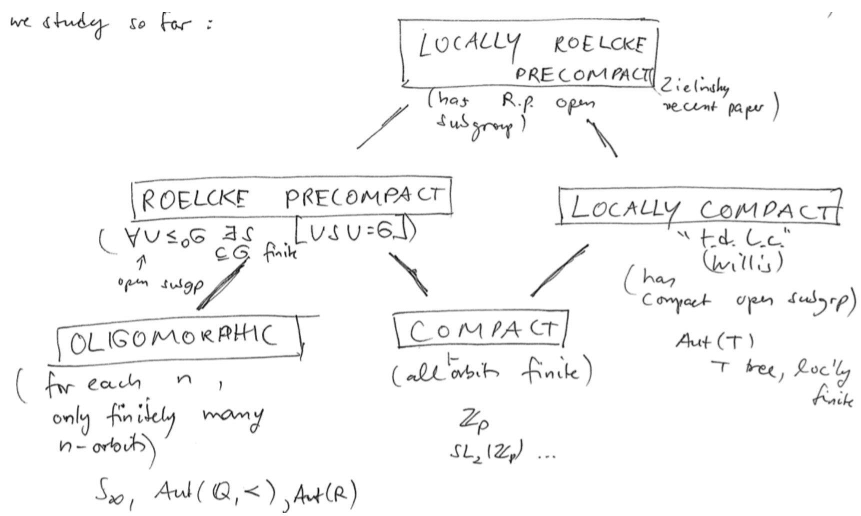
Example

- Every compact (i.e., profinite) subgroup of S_∞ is R.p. For, compactness means that U has only finitely many cosets.
- Every oligomorphic group (i.e., automorphism group of an ω-categorical structure) is R.p.

Definition (Zielinski, recent preprint)

A closed subgroup G of S_∞ is called locally Roelcke precompact if it has an open R.p. subgroup. E.g., the totally disconnected locally compact groups are locally R.p.

Classes of closed subgroups of S_∞ under inclusion



General duality theorem

Theorem (N., Schlicht and Tent 2021; Melnikov and N.)

- Let $\mathcal{C}$ be a class in the diagram.
- For each $G \in \mathcal{C}$, let $\mathcal{W}(G)$ be its approximation groupoid, a countable structure to be defined on the R.p. open cosets in G.
- Let $\mathcal{D}$ be the closure under isomorphism of the range of $\mathcal{W}$ (among closed subgroups of S_∞).

THEN:

- $\mathcal{D}$ is a Borel class.
- $\mathcal{W}$ is a Borel map, and there is a Borel map $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ so that $\mathcal{W}$ and $\mathcal{G}$ are inverses up to isomorphism:
  For each $G \in \mathcal{C}$ and each $M \in \mathcal{D}$,
  $$\mathcal{G}(\mathcal{W}(G)) \cong G \text{ and } \mathcal{W}(\mathcal{G}(M)) \cong M.$$

Consequences of the theorem

The theorem says in brief that there is an operator $\mathcal{W}$ on the locally R.p. groups so that for each class $\mathcal{C}$ in the diagram,

◦ the closure under isomorphism of the range of $\mathcal{W} \restriction \mathcal{C}$ is Borel, and

◦ $\exists$ a Borel map $\mathcal{G}$ so that $\mathcal{W}$ and $\mathcal{G}$ are inverses up to isomorphism.

- In all cases except for $\mathcal{C}$ = oligomorphic, the proof provides a category equivalence for the categories with objects the structures in $\mathcal{C}$ and $\mathcal{D}$, and morphisms the isomorphisms.
- NST '21 used the Borel duality to give an upper bound on the complexity of $\cong$ on the oligomorphic groups: it is $\leq_B$ a Borel equivalence relation with all classes countable.
- Melnikov and N. (in prep.) use an effective version of this duality to define when a group in $\mathcal{C}$ is computable, where $\mathcal{C}$ consists of the locally compact subgroups of S_∞.

Define approximation groupoids axiomatically

Let G be locally Roelcke precompact. We collect properties of the set of open R.p. cosets of G.
A, B, C denote these cosets
U, V, W denote open R.p. subgroups of G.

- An approximation groupoid is a structure $(M, \circ, \sqsubseteq, \emptyset)$ s.t.
- $(M, \circ)$ is a small category where each morphism has an inverse. $A \colon U \to V$ means that

  A is a right coset of U and a left coset of V.
- $(M, \sqsubseteq, \emptyset)$ is a lower semilattice with least element $\emptyset$.
- Note that $\emptyset$ is added only for convenience. A, B, C, U, V, W always denote elements of $M - \{\emptyset\}$.

Note: if $A \colon U \to V$ then $A^{-1} \colon V \to U$.
If $A \colon U \to V$ and $B \colon V \to W$ then $A \circ B \colon U \to W$.

AG is a structure $(M, \circ, \sqsubseteq, \emptyset)$. $(M, \circ)$ is a small category. Identify object U with morphism 1_U. Write AB for $A \circ B$. $(M, \sqsubseteq, \emptyset)$ is a l.s.l. with least element $\emptyset$. By $A \perp B$ denote that $A \wedge B = \emptyset$. Write ${}_U A$ to denote that $UA = A$, ie A is right coset of U.

(A1) If $A \sqsubseteq B$ then $A^{-1} \sqsubseteq B^{-1}$.
(A2) Let $U \sqsubseteq V$.
  ($\downarrow$) If ${}_V B$ then $A \sqsubseteq B$ for some ${}_U A$.
  ($\uparrow$) If ${}_U A$ then $A \sqsubseteq B$ for some ${}_V B$.
(A3) if AB and $A'B'$ are defined and $A \sqsubseteq A', B \sqsubseteq B'$, then $AB \sqsubseteq A'B'$.
(A4) If ${}_U A$ and ${}_V B$ and $U \sqsubseteq V$, then either $A \sqsubseteq B$ or $A \perp B$.
(A5) If $A \not\sqsubseteq B$ then there is $C \sqsubseteq A$ such that $C \perp B$.

Axioms (A1), (A2$\downarrow$) and (A3) are known as the axioms of ordered groupoids; see e.g. Lawson's 1998 book "Inverse semigroups".

Example of an approximation groupoid

Consider the oligomorphic group $G = \mathrm{Aut}(\mathbb{Q}, <)$. The open subgroups of G are the stabilizers of finite sets.

- If U, V are stabilizers of sets of the same finite cardinality, there is a unique morphism $A \colon U \to V$ in the sense above, corresponding to the order-preserving bijection between the two sets.
- The approximation groupoid for $\mathrm{Aut}(\mathbb{Q}, <)$ is canonically isomorphic to the groupoid of finite order-preserving maps on $\mathbb{Q}$, with the partial order being reverse extension.
- If maps A, B are compatible, the meet $A \wedge B$ is the union of those maps. Else the meet is $\emptyset$.

Proving the Borel duality theorem given a class $\mathcal{C}$ in the diagram (some ideas)

We have already defined the Borel map $\mathcal{W} \colon \mathcal{C} \to \mathcal{D}$, where $\mathcal{D} = [\mathrm{range}(\mathcal{W})]_{\cong}$. Next:

(a) ensure that $\mathcal{D}$ is Borel by adding axioms that depend on $\mathcal{C}$.
(b) Define the Borel inverse map $\mathcal{G} \colon \mathcal{D} \to \mathcal{C}$.

Definition (Melnikov and N.)

Given an approximation groupoid M with domain $\mathbb{N}$, let $\mathcal{G}(M)$ be the closed subgroup of S_∞ consisting of the functions p such that

- p preserves $\sqsubseteq$, and
- if $A \circ B$ is defined then $p(A) \circ B = p(A \circ B)$.

Proving the Borel duality theorems (2)

That $\mathcal{G}(\mathcal{W}(G)) \cong G$ was essentially shown already in Kechris, N. and Tent, 2018, but with a somewhat different definition of the inverse operator $\mathcal{G}$.
For each class $\mathcal{C}$ in the diagram, some additional axioms are posited for an approximation groupoid M:

- "$\mathcal{G}(M) \in \mathcal{C}$" to ensure that $\mathcal{D}$ is Borel. (This is a bit silly, and in the cases except for (locally) Roelcke precompact, ' we know how to express it more concretely by a $L_{\omega_1,\omega}$ axiom, and in some cases f.o .)
- an axiom ensuring that each R.p. open coset $\mathcal{R}$ in $\mathcal{G}(M)$ corresponds to an element A of M, i.e.,
$$\mathcal{R} = \widehat{A} = \{p \in \mathcal{G}(M) \colon p(U) = A\}$$
for some left coset A of some $U \in M$. This implies that $\mathcal{W}(\mathcal{G}(M)) \cong M$ for each $M \in \mathcal{C}$.

Borel duality for compact subgroups of S_∞.

In the following, given an abstract approximation groupoid, by a *subgroup we mean an element of M that behaves like a subgroup, i.e. $U \circ U = U$. Similarly we apply * to other terms.
Given an approximation groupoid M and a *subgroup $U \in M$, let $LC(U)$ denote the left *cosets of U, i.e.
$LC(U) = \{A \in M \colon\ AU = A\}$. Similarly define $RC(A)$.

Definition

An approximation groupoid M is *compact if
$M \models \forall U\,[LC(U) \text{ is finite}]$.

Proposition

Let M be an approximation groupoid. Then
M is *compact $\iff \mathcal{G}(M)$ is compact.

Axiom CC (completeness in presence of compactness).

Let M be a *compact approximation groupoid.
Let N be a *subgroup in M such that $LC(N) = RC(N)$.
(I.e., N is a normal *subgroup.)
If a set $\mathcal{S} \subseteq LC(N)$ is closed under products and inverses, then there is a *subgroup U such that $A \sqsubseteq U \leftrightarrow A \in \mathcal{S}$, for each $A \in LC(N)$.

Note that if G is profinite then $\mathcal{W}(G)$ satisfies this axiom.

Proposition

Let M be a *compact approximation groupoid satisfying Axiom CC. Then $M \cong \mathcal{W}(\mathcal{G}(M))$ via the map

$$A_U \mapsto \widehat{A} = \{p \in \mathcal{G}(M) \colon p(U) = A\}.$$

Totally disconnected locally compact groups

Van Dantzig's theorem from the 1920s says that each tdlc group has a compact open subgroups. T.d.l.c. groups are frequently studied since the 1990, e.g. by George Willis and his co-workers. Some of the work parallels the study of Lie groups.
There are important native notions, such as the scale function
$\sigma(x) = \{\min |V \colon x^{-1}Vx \cap V| \colon V \text{ is compact open subgroup}\}$.
Examples of tdlc groups:

- $G = (\mathbb{Q}_p, +)$, the p-adics for a prime p. The proper open subgroups are compact, and are all of the form $p^r\mathbb{Z}_p$ for some $r \in \mathbb{Z}$.
- $\mathrm{Aut}(T_d)$ for $d \geq 2$. This is the group of automorphism of the d-regular tree T_d, defined as an undirected graph without a specified root. Each proper open subgroup is compact. Each compact subgroup is contained in the stabilizer of a vertex, or the stabilizer of an edge.

Effectively locally compact subtrees of $\omega^{<\omega}$

We work towards a definition of computable tdlc groups.
Let $T \subseteq \omega^{<\omega}$ be a computable tree without dead ends. Let $[T] \subseteq \omega^\omega$ be the set of paths. For $\sigma \in T$ let

$$[\sigma] = \{X \in [T] \colon \sigma \prec X\}.$$

We call $[T]$ effectively locally compact if $[T]$ is locally compact, and

- given a string $\sigma \in T$ one can decide whether $[\sigma] \cap T$ is compact;
- further, in that case one can uniformly compute a function h such that if $\rho \in T$ extends σ, then $\rho(i) < h(i)$ for each i.

- Note that each compact open subset K of $[T]$ has the form $K = \bigcup_{\sigma \in V} [\sigma]$ for some finite set of strings V.
- If $K = \bigcup_{\sigma \in D_e} [\sigma]$ we call e a strong index for K.

Definition

A computable presentation of a totally disconnected Polish group G is a computable tree $T \subseteq \omega^{<\omega}$ without dead ends, and operations

$$\mathrm{Op} \colon [T] \times [T] \to [T] \text{ and } \mathrm{Inv} \colon [T] \to [T]$$

that are effectively open and effectively continuous (i.e., given by Turing functionals), such that $G \cong ([T], \mathrm{Op}, \mathrm{Inv})$.

Definition

A computable tdlc presentation of a tdlc group G is a computable presentation $([T], \mathrm{Op}, \mathrm{Inv})$ of G which additionally satisfies:

- $[T]$ is effectively locally compact, and
- given $\sigma, \tau \in T$ such that $[\sigma], [\tau]$ are compact, one can compute strong indices for the compact open sets $\mathrm{Inv}[(\sigma)]$ and $\mathrm{Op}([\sigma], [\tau])$.

Characterization of computable tdlc groups in terms of their approximation groupoids

Given an approximation groupoid M, we say that a $*$-subgroup $U \in M$ is $*$-compact if the subgroupoid induced on $\{B \colon B \sqsubseteq U\}$ is $*$-compact.

Theorem (Melnikov and N.)

For a Polish tdlc group G, the following are equivalent:

- G has a computable tdlc presentation.
- the approximation groupoid of G has a computable copy with an algorithm for $*$-compactness.

The theorem holds uniformly.
Since the second condition can be checked to apply to the tdlc groups $\mathbb{Q}_p$ and $\mathrm{Aut}(T_d)$, they have computable presentations.

Computable presentation of S_∞

For strings σ_i, $i = 0, 1$, of the same length N, we write $\sigma_0 \oplus \sigma_1$ for the string σ of twice that length such that $\sigma(2i + k) = \sigma_k(i)$ for each $k \leq 1$ and $i < N$. Define a tree without dead ends by $\mathbb{S} = \{\sigma \oplus \tau \in \omega^{<\omega} \colon$

$$|\sigma| = |\tau| \,\wedge\, \sigma, \tau \text{ are 1-1} \,\wedge\, \forall i, k < |\sigma| \, [\sigma(i) = k \leftrightarrow \tau(k) = i]\}.$$

The paths of $\mathbb{S}$ consist of the permutations of ω, paired with their inverses:

$$[\mathbb{S}] = \{f \oplus f^{-1} \colon f \in S_\infty\}.$$

Inv and Op in the computable presentation of S_∞

Recall: A computable presentation of a totally disconnected Polish group G is a computable tree $T \subseteq \omega^{<\omega}$ without dead ends, and operations $\mathrm{Op}\colon [T] \times [T] \to [T]$ and $\mathrm{Inv}\colon [T] \to [T]$ that are effectively open and effectively continuous, and such that $G \cong ([T], \mathrm{Op}, \mathrm{Inv})$.

- Let Inv_{S_∞} be the Turing functional such that $\mathrm{Inv}_{S_\infty}(f_0 \oplus f_1) = f_1 \oplus f_0$ for any functions f_0, f_1.
- The Turing functional for the operation Op_{S_∞} is determined by taking the (right-bound) composition of two elements, and their inverses: $\mathrm{Op}_{S_\infty}(f_0 \oplus f_1, g_0 \oplus g_1) = f_0 g_0 \oplus g_1 f_1)$.

Fact

$(\mathbb{S}, \mathrm{Op}_{S_\infty}, \mathrm{Inv}_{S_\infty})$ is a computable presentation of S_∞.
In particular, Inv_{S_∞} and Op_{S_∞} are effectively open.

Compactness preservation is effective for computable closed subgroups of S_∞

- Suppose we are given a computable subtree without dead ends T of $\mathbb{S}$ such that $[T]$ is a subgroup of S_∞.
- Given $\sigma, \tau \in T$ such that $[\sigma], [\tau]$ are compact, one can compute strong indices for the compact open sets $\mathrm{Inv}_{S_\infty}[(\sigma)]$ and $\mathrm{Op}_{S_\infty}([\sigma], [\tau])$.
- So the second condition in the definition of computable tdlc presentation (Slide 22) is satisfied automatically.

Computable tdlc presentations on S_∞

- It's well known that each tdlc group is isomorphic to a closed subgroup of S_∞.
- We show that each computable tdlc presentation leads to a computable tdlc presentation via a computable subtree T of $\mathbb{S}$.

Theorem (Melnikov and N.)

Suppose G has a computable tdlc presentation.
Then there is a computable subtree T of $\mathbb{S}$ without dead ends such that $[T]$ together with the restrictions of Inv_{S_∞} and Op_{S_∞} to $[T]$ forms a computable tdlc presentation of G.

The proof is uniform. It applies the previous theorem: we construct $[T]$ from the approximation groupoid $\mathcal{W}(G)$, which has a computable copy with an algorithm for $*$-compactness.

Computable tdlc groups

For an abelian locally compact group G, the Pontryagin-van Kampen dual $\widehat{G}$ is the topological group of characters of G.

Theorem (Lupini, Melnikov and N., submitted)

Suppose G is an abelian tdlc group.
If G is computable then $\widehat{G}$ is computably metrized Polish.

If $\widehat{G}$ is tdlc, then the converse holds.

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