Randomness and initial segment complexity for probability measures

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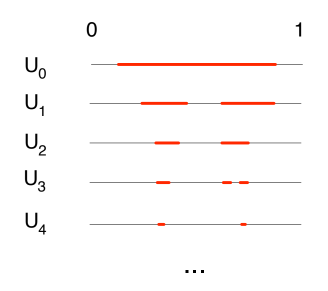
Martin-Löf's randomness notion (1966)

- A Martin-Löf test is a uniformly recursively enumerable sequence $\langle U_m \rangle_{m \in \mathbb{N}}$ of open sets in the space $\{0,1\}^{\mathbb{N}}$ of infinite bit sequences such that $\lambda U_m \leq 2^{-m}$ for each m. Here λ is the uniform (product) measure giving both 0 and 1 the same probability.
- A bit sequence Z is Martin-Löf random if Z passes each ML-test, in the sense that Z is only in finitely many of the U_m.



Example of a ML-test. (We picture open sets as unions of intervals via binary representations.)

Characterization of ML-randomness via the initial segment complexity

Let $K(x)$ be the length of a shortest prefix free description of a binary string z. Given $Z \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $Z \upharpoonright n$ denote the initial segment $Z(0) \ldots Z(n-1)$.

Schnorr's Theorem informally says that

Z is ML-random $\iff$ each initial segment of Z is incompressible.

Theorem (Schnorr 1973)

Z is ML-random $\iff$ there is $b \in \mathbb{N}$ such that $\forall n\, K(Z \upharpoonright n) \geq n - b$.

Levin (1973) proved the analogous theorem for monotone string complexity.

Generalise to measures?

A probability measure μ on $\{0,1\}^{\mathbb{N}}$ can be seen as a statistical superposition of infinite bit sequences.

How do we define algorithmic randomness for such superpositions?

Plan

- We plan to study algorithmically defined randomness properties of probability measures μ on Cantor space $\{0,1\}^{\mathbb{N}}$.
- This should generalise the case of infinite bit sequences, which correspond to Dirac measures.
- We define initial segment complexity for measures.
- We relate the growth of initial segment complexity to randomness properties.

Randomness for infinite sequences of qubits

- Nies and Scholz, J. Math. Physics, 2019, introduced ML-randomness for infinite “sequences” of quantum bits;
- these sequences are states ρ on a C^*-algebra M_∞ from physics called the “CAR algebra”.
- They can be viewed as coherent sequences of density matrices $D_n \in M_{2^n}$. Here M_{2^n} is the algebra of $2^n \times 2^n$ matrices over $\mathbb{C}$. Coherence means that the partial trace of D_{n+1}, wrt last qubit, is D_n.
- $D_n = \rho \upharpoonright n$ is the n-th initial segment.
- Probability measures can be seen as states where all the matrices are diagonal.
- Our definition of randomness for measures is a special case of theirs by recent work of Bhojraj.

Martin-Löf absolutely continuous measures

Recall that a measure μ is continuous if it has no atoms, and absolutely continuous if $\mu(\mathcal{N}) = 0$ for each λ-null set $\mathcal{N}$.

Definition (Main)

A measure μ on $\{0,1\}^{\mathbb{N}}$ is called Martin-Löf absolutely continuous (ML-a.c., for short) if

$$\inf_m \mu(G_m) = 0 \text{ for each Martin-Löf-test } \langle G_m \rangle.$$

Fact

- The uniform measure λ is ML-a.c..
- Let $\mu = \sum_k c_k \delta_{Z_k}$ be a positive sum of Dirac measures. Then μ is ML-a.c. $\iff$ all Z_k are Martin-Löf-random.
- In particular δ_Z is ML-a.c. for each ML-random Z.

Definition (Recall)

A measure μ on Cantor space is called Martin-Löf absolutely continuous (ML-a.c., for short) if $\inf_m \mu(G_m) = 0$ for each Martin-Löf-test $\langle G_m \rangle$.

- It suffices to consider descending Martin-Löf-tests, because we can replace $\langle G_m \rangle$ by the Martin-Löf-test $\widehat{G}_m = \bigcup_{k>m} G_k$.
- So we can change the passing condition to $\lim_m G_m = 0$.

Since there is a universal ML-test, μ being ML-a.c. simply means that

$$\mu(\text{non-MLR}) = 0.$$

Solovay tests

The following test notions appears to be more general than ML-tests, but it in fact isn't.

- A Solovay test is a sequence $\langle S_n \rangle$ of uniformly Σ^0_1 sets such that $\sum_k \lambda S_k < \infty$. (Before, we required $\lambda S_k \leq 2^{-k}$.)
- A measure μ passes such a test if $\lim_k \mu(S_k) = 0$.

Proposition

A measure μ is ML-a.c. $\iff$ μ passes each Solovay test.

Proof:

- by the known equivalence for bit sequences, the set $\limsup_k S_k = \{Z \colon \exists^\infty k\, [Z \in S_k]\}$ only consists of non-MLR sequences.
- So $\mu(\limsup_k S_k) = 0$.

Descriptive complexity of initial segments for measures

x will denote a (finite) bit string.

- Let $C(\mu \upharpoonright n) = \sum_{|x|=n} C(x)\mu[x]$ be the μ-average of all the $C(x)$ over all strings x of length n.
- In a similar way we define $K(\mu \upharpoonright n)$.

Fact

Let λ denote the uniform measure. We have $C(\lambda \upharpoonright n) \geq n - 1$.

Pf.: $C(\lambda \upharpoonright n) = \sum_{r=0}^{n+d} \sum_{x:|x|=n \wedge C(x) \geq r} 2^{-n} \geq$
$\sum_{r=0}^{n} [\sum_{|x|=n} 2^{-n} - \sum_{|x|=n, C(x)<r} 2^{-n}] \geq n+1 - \sum_{r \leq n} 2^{-n+r} \geq n-1.$

Both implications of Schnorr's theorem fail

We say that μ has complex initial segments if $K(\mu \upharpoonright n) \geq^+ n$. The analog of Levin-Schnorr fails for measures in both directions.

Proposition

There is a ML-a.c. measure μ such that for each $\theta \in (0, 1)$, $K(\mu \upharpoonright n) \leq^+ n - n^\theta$. ($\leq^+$ means up to a constant.)

We falsify the converse implication of L-S by the following.

Theorem

There are a ML random bit sequence X and a non-ML random Y such that, for all n, $K(X \upharpoonright n) + K(Y \upharpoonright n) \geq^+ 2n$.

Corollary

The measure $\mu = \frac{1}{2}(\delta_X + \delta_Y)$ has complex initial segments but is not ML-a.c.

Strong Chaitin randomness for measures

Recall that a measure μ on Cantor space is called Martin-Löf absolutely continuous if $\inf_m \mu(G_m) = 0$ for each Martin-Löf-test $\langle G_m \rangle$.

$Z \in \{0,1\}^{\mathbb{N}}$ is called strongly Chaitin random if there is $d \in \mathbb{N}$ such that $K(Z \upharpoonright n) \geq n + K(n) - d$ for infinitely many n.

This is equivalent to ML-randomness relative to the halting problem by Miller (2010).

Theorem

Suppose that for some r, we have $K(\mu \upharpoonright n) \geq n + K(n) - r$ for infinitely many n. Then μ is ML-a.c.
Similar under the hypothesis $C(\mu \upharpoonright n) \geq n - r$ for infinitely many n.

The uniform measure on the space of measures

- The "uniform" probability measure $\mathbb{P}$ on the space $\mathcal{M}(\{0,1\}^{\mathbb{N}})$ of probability measures on Cantor space has been studied by Mauldin and Monticino (1995).
- Idea: if $\mu[x]$ has been determined for a string x, choose $\mu[x0] \leq \mu[x]$ uniformly at random.
- The algorithmic theory has been developed in Culver's PhD thesis (Notre Dame, 2015). μ is ML-random wrt $\mathbb{P} \Rightarrow \mu$ is continuous.

Proposition

μ is Martin-Löf-random wrt to $\mathbb{P} \Rightarrow \mu$ is ML-a.c.

- $\lambda(G) = \int \mu(G) d\mathbb{P}(\mu)$ for each open G.
- Use this to show: if $\langle G_m \rangle$ is a Martin-Löf-test such that $\inf_m \mu(G_m) > 0$, then μ is not Martin-Löf-random w.r.t. $\mathbb{P}$.

K-triviality for measures

Definition

A measure μ is called K-trivial if $\exists b \forall n \, K(\mu \restriction n) \leq K(n) + b$.

For Dirac measures δ_A this means that A is K-trivial in the usual sense.

Theorem

Suppose μ is K-trivial. Then μ is concentrated on its atoms.

Use that $\forall c \; \exists d \; \forall n$

$$\exists^{\leq d} \text{ strings } x \text{ of length } n \text{ with } K(x) \leq K(n) + c.$$

Ergodic measures

- T denotes the shift operator on $\{0,1\}^{\mathbb{N}}$. A measure ρ is shift-invariant if $\rho(A) = \rho(T^{-1}(A))$ for each Borel A.
- A shift-invariant measure ρ is ergodic if every ρ-integrable function f with $f \circ T = f$ is constant ρ-a.s.
- For ergodic ρ, the entropy $H(\rho)$ is defined as $\lim_n H_n(\rho)$, where
$$H_n(\rho) = -\frac{1}{n} \sum_{|w|=n} \rho[w] \log \rho[w].$$
- $H(\lambda) = 1$ and this is the maximum possible entropy,

SMB Theorem (1950s)

For $n \geq 0$, for $Z \in \{0,1\}^{\mathbb{N}}$ let $h_n^\rho(Z) = -\frac{1}{n} \log \rho[Z \upharpoonright n]$.

This is the weighted log-likelihood random variable.
Note that $H_n(\rho) = E_\rho h_n^\rho$ where E_ρ denotes the expectation w.r.t. ρ.

Theorem (Shannon-McMillan-Breiman theorem)

Let ρ be an ergodic measure.
For ρ-a.e. $Z \in \{0,1\}^{\mathbb{N}}$ we have $\lim_n h_n^\rho(Z) = H(\rho) := \lim_n H_n(\rho)$.

Algorithmic version:
If ρ is computable, then the conclusion holds for ρ-ML-random Z by results of Hochman (2009, implicit) and Hoyrup (2012).

Recall: for $n \geq 0$, for $Z \in \{0,1\}^{\mathbb{N}}$ let $h_n^\rho(Z) = -\frac{1}{n} \log \rho[Z \upharpoonright n]$.

We say that a measure μ is ML-a.c. relative ρ if $\mu(G_m) \to 0$ for each ρ-ML test $\langle G_m \rangle$.

Proposition (Effective SMB theorem for measures)

Let ρ be a computable ergodic measure. Suppose that there is D such that $h_n^\rho \leq D$ for each n. If μ is ML-a.c. w.r.t. ρ then $\lim_n E_\mu h_n^\rho = H(\rho)$, where $E_\mu h_n^\rho = \sum_{|x|=n} h_n^\rho(x)\mu([x])$.

Example (Boundedness condition is necessary)

Let $p_k = 2^{-k^4}/c$ where c is chosen so that $\sum_{k \geq 1} p_k = 1$. Let ρ be the measure associated with the corresponding binary renewal process: $\rho(10^k 1 \prec Z \mid Z_0 = 1) = p_k$ for each $k \geq 1$.
Then there is a computable measure $\mu \ll \rho$ such that $E_\mu h_n^\rho \to \infty$.

A measure version of Brudno's Theorem

The weighted asymptotic initial segment complexity of a ML-a.c. measure relative to ρ obeys some positive lower bound.

Proposition

Let ρ be a computable ergodic measure, and suppose μ is a Martin-Löf a.c. measure with respect to ρ. Then

$$\lim_n \frac{1}{n} K(\mu \upharpoonright n) = \lim_n \frac{1}{n} C(\mu \upharpoonright n) = H(\rho).$$

Recall that we obtained a ML-a.c. measure μ such that $K(\mu \upharpoonright n) \leq^+ n - n^\theta$ for any $\theta \in (0,1)$. Since $H(\lambda) = 1$, we now see that we cannot subtract, say, $n/4$ instead of n^θ.

Summary and open questions

$$\begin{tikzcd}
\mu \text{ is a.c.} \arrow[ddr] & \forall n\, C(\mu \restriction n) \geq^+ n \arrow[d] & \mu \text{ is MLR w.r.t. } \mathbb{P} \arrow[ddl] \\
 & \exists^\infty n\, C(\mu \restriction n) \geq^+ n \arrow[d] & \\
 & \mu \text{ is ML-a.c.} \arrow[d] & \\
 & \lim_n \frac{1}{n} C(\mu \restriction n) = 1 &
\end{tikzcd}$$

Culver: μ MLR w.r.t. $\mathbb{P} \Rightarrow \mu$ is orthogonal to λ, and hence not a.c. The middle column of the diagram is strict, via examples that are Dirac measures.

Future research: improve the diagonal arrows in the diagram.

Study the case of randomness relative to a general ergodic computable ρ.

THANKS!

References:

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