Closure of resource bounded randomness notions under polynomial time permutations

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A hierarchy of randomness notions

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Martin-Löf rd. $\Rightarrow$ computably random $\Rightarrow$ Schnorr random

Algorithmic tests in the resource–bounded setting

A martingale M is a function from $\{0,1\}^*$ to $\{q \in \mathbb{Q} : q > 0\}$ satisfying

$$M(x) = \frac{M(x0) + M(x1)}{2} \text{ for all } x \in \{0,1\}^*.$$

M succeeds on a bit sequence Z if $\limsup_n M(Z \upharpoonright n) = \infty$.

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M succeeds on a bit sequence Z if $\limsup_n M(Z \upharpoonright n) = \infty$.

- A martingale M is polynomial time computable if on input x one can determine the rational $M(x)$ in polynomial time.
- Here a positive rational number q is represented by the pair $\langle k, n\rangle$ of natural numbers (written in binary) such that $q = k/n$ in lowest terms.
- Similarly, define when a martingale is polynomial space computable.

Polynomial time and poly space randomness

- $Z \in \{0,1\}^{\mathbb{N}}$ is polynomial time random if no polynomial time computable martingale succeeds on Z.
- This was briefly defined by Schnorr (1971). Lutz (1990) studied resource bounded martingales. Ambos-Spies and Mayordomo looked at the associated randomness notions (for sets of strings, rather than bit sequences). Polynomial randomness was studied in more explicit form in Yongge Wang's 1996 thesis (Uni Heidelberg)[a].

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- In a similar way one defines polynomial space randomness.

[a]See the 1996 survey "Resource bounded measure and randomness" by Ambos-Spies and Mayordomo for background and references.

Existence result

From now on we identify a bit sequence $Z \in \{0,1\}^{\mathbb{N}}$ with a subset of $\{0\}^*$ (a tally language):

$Z \in \{0,1\}^{\mathbb{N}}$ is seen as $\{0^k \colon Z(k) = 1\}$.

In this way we can apply the notions of complexity theory to Z.

Fact

Polytime randoms exist in all superpolynomial time classes.

E.g, there is one in $\mathsf{DTIME}(n^{\log n})$.

Base invariance of polynomial time randomness

- Polynomial time martingales and polytime randomness in base $b > 2$ are defined similar to the above.
- Polynomial time randomness can be seen as a property of real numbers, rather than of sequences of digits for a fixed base:

Theorem (Figueira, N 2013) For a real number $r \in [0, 1]$,

- let Z be the binary expansion of r
- let Y be the expansion of r in base b.

Z is polynomial time random $\iff$ Y is polynomial time random.

A similar result was previously known for computable randomness. In both cases, one uses a characterization of the randomness notion for reals via differentiability of certain effectively computable functions.

Permutations with polynomially unbounded inverse

Proposition. Let S be a polynomial time computable permutation of $\{0\}^*$ such that for each polynomial p, there are infinitely many n with $p(S(n)) \leq n$.

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There is a polynomial time random $Z \subseteq \{0\}^*$ (computable in time $2^{O(n)}$) such that $Z \circ S$ is not polynomial time random.

It is not hard to see that a permutation S as in the Proposition exists. Hence one should only look for closure under polynomial time computable permutation S such that S^{-1} is polynomially bounded.

Part 1: If P= PSPACE then closure holds

Closure of polynomial space randomness 1

Based on the arguments in Buhrman, Melkebeek, Regan, Sivakumar and Strauss (2000), we show that polynomial space randomness is closed under polynomial time computable permutations S with polynomially bounded inverse.

So if $\mathsf{P} = \mathsf{PSPACE}$, this closure property applies to polynomial time randomness as well.

- Let $g(n)$ be a polynomial bound for S^{-1}.
- Suppose that a polynomial space martingale B succeeds on $Z \circ S$. We may assume that B has the savings property:

$$\text{if } \sigma \preceq \tau \text{ then } B(\tau) \geq B(\sigma) - 2.$$

Closure of polynomial space randomness 2

The "savings – gale" $B \in \mathsf{PSPACE}$ succeeds on $Z \circ S$.

We define a martingale $D \in \mathsf{PSPACE}$ that succeeds on Z.

For bit strings α, w such that $|\alpha| = g(|w|)$, we write $\alpha \sim_S w$ if $\alpha = w \circ S$ whenever both sides are defined, namely

$$k < |\alpha| \wedge S(k) < |w| \Rightarrow \alpha(k) = w(S(k)).$$

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There are $2^{g(|w|)-|w|}$ many α's of length $g(|w|)$ such that $\alpha \sim_S w$. To define $D(w)$ we take their average:

$$D(w) = 2^{|w|-g(|w|)} \sum B(\alpha) \, [\![|\alpha| = g(|w|) \ \wedge \ \alpha \sim_S w]\!]$$

Part 2: If BPP is large then closure fails

The class BPP

- BPP is short for “bounded-error probabilistic polynomial time”.
- BPP contains the languages for which there is a polynomial time algorithm that uses random bits and obtains the answer with error probability $< 1/2 - \epsilon$, for some fixed ϵ.
- By repeating runs independently, we can ensure that the error probability on inputs of length n is at most $2^{-q(n)}$ for a given polynomial q.

Example of a problem in BPP that is not known to be in P: polynomial identity testing. Polynomials are given as arithmetical circuits, and we have to test whether they are equal.

If BPP contains a time class larger than P then closure fails

Theorem. Assume that $\mathsf{DTIME}(h) \subseteq \mathsf{BPP}$ for some time constructible function h that eventually dominates each polynomial, e.g. $n^{\log n}$. Then there is a polynomial time random set $Z \in \mathsf{DTIME}(2^{3 \cdot n})$ and a permutation S with $S, S^{-1} \in \mathsf{P}$ such that

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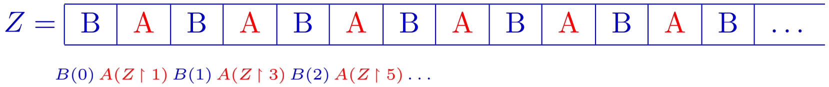
We may assume that $h(n) \leq n^{\log n}$.

Let M be a martingale in $\mathsf{DTIME}(h)$ that bets only on odd positions, and dominates up to a multiplicative constant all polynomial time martingales that bet only on odd positions. Let

$$A = \{x \in \{0,1\}^* : x \text{ has odd length and } M(x1) < M(x0)\}.$$

A is computable in time h and hence by assumption in BPP.
Let $B \subseteq \{0\}^*$ be a set in $\mathsf{DTIME}(2^{5 \cdot n})$ so that no martingale in $\mathsf{DTIME}(2^{4 \cdot n})$ succeeds along B. Define $Z \subseteq \mathbb{N}$ as follows:

$$Z(2n) = B(n);\; Z(2n+1) = A(Z \upharpoonright 2n+1).$$



Z is polynomial time random

$Z =$	B	A	B	A	B	A	B	A	B	A	B	A	B	...

$B(0)\ A(Z \upharpoonright 1)\ B(1)\ A(Z \upharpoonright 3)\ B(2)\ A(Z \upharpoonright 5) \ldots$

Assume that a polynomial time martingale L succeeds on Z. We may suppose L bets only on odd positions (the A's), or only on even positions (the B's).

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$B(0)$ $A(Z \upharpoonright 1)$ $B(1)$ $A(Z \upharpoonright 3)$ $B(2)$ $A(Z \upharpoonright 5) \dots$

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- The case "odd positions" cannot happen: by definition of A, the capital of the universal M, and hence of L, is bounded along Z.

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- In the case "even positions", we define a martingale $N \in \mathsf{DTIME}(nh(n) + h(n))$ that succeeds on B, contradiction.

Z is polynomial time random

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- The case "odd positions" cannot happen: by definition of A, the capital of the universal M, and hence of L, is bounded along Z.
- In the case "even positions", we define a martingale $N \in \mathsf{DTIME}(nh(n) + h(n))$ that succeeds on B, contradiction.
- N computes the values of Z at the even positions using that $A \in \mathsf{DTIME}(h)$. It doesn't need to bet on these values; they are only needed to determine the bets of L at the odd positions.

A reshuffling $\widehat{Z}$ of Z

Since $A \in \mathsf{BPP}$, there is a randomised algorithm $\mathcal{R}$ that computes $A(x)$ on input $x \in \{0,1\}^{2n+1}$ with error probability 2^{-3n-2}. This takes time $p(n)$ for some polynomial p, and hence uses at most $p(n)$ random bits.

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Let $\widehat{Z}$ consisting for $n = 0, 1, \ldots$ of $p(n)$ bits taken from B, followed by the bit $A(Z \restriction 2n+1)$.

$$\widehat{Z} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c} \hline \mathrm{B} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{A} & \ldots \\ \hline \multicolumn{1}{c}{p(0)} & \multicolumn{1}{c}{} & \multicolumn{3}{c}{p(1)} & \multicolumn{1}{c}{} & \multicolumn{6}{c}{p(2)} & \multicolumn{1}{c}{} & \end{array}$$

Then $\widehat{Z} = Z \circ S$ for the right permutation S, and $S, S^{-1} \in \mathsf{P}$.

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Idea: use B as a reservoir of random bits.

The probability that a string y of length $p(n)$ miscomputes $A(x)$ for some x of length $2n+1$ is at most $2^{2n+1} \cdot 2^{-3n-2} = 2^{-n-1}$. Whether y miscomputes some x can be determined in time $O(2^{3n})$.

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Starting with 2^{-n-1} of the initial capital set aside for this purpose, we can bet on the set of miscomputing y. On each such y we gain EUR 1. If the portion of B's of length $p(n)$ is miscomputing y i.o. then the martingale succeeds on the sequence B.

The reshuffling $\widehat{Z}$ is not polynomial time random

$\widehat{Z} =$	B	A	B	B	B	A	B	B	B	B	B	B	A	$\ldots$
	$p(0)$			$p(1)$						$p(2)$				

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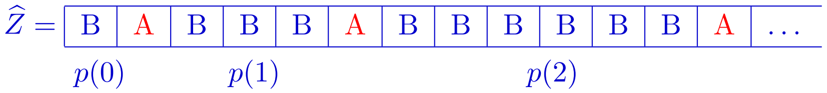

- The martingale explained above can be computed in time $O(2^{4k})$.
- This uses that $p(n+1) - p(n) \leq p(n)$ for a.e. n. For $k = p(n)$ we have to average over at most 2^k many y's satisfying a condition that can be checked in time $O(2^{3n})$. This yields the exponent 4.

Since B is $O(2^{4n})$-random, almost every such portion of B's computes the value of $\widehat{Z}$ at the "A" that comes after correctly.

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p(0) & & & p(1) & & & & & & p(2) & & & &
\end{array}$$

To repeat, almost every such portion of B's of length $p(n)$ computes the value of $\widehat{Z}$ at the "A" that comes after correctly.

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	$p(0)$			$p(1)$						$p(2)$				

To repeat, almost every such portion of B's of length $p(n)$ computes the value of $\widehat{Z}$ at the "A" that comes after correctly.
We use this to define a polynomial time martingale that succeeds on $\widehat{Z}$.
To make bet at the position r_n of the n-th "A":

- From $\widehat{Z} \upharpoonright r_n$ determine $x = Z \upharpoonright 2n + 1$
- let $y =$ the portion of B's preceding the n-th "A"
- run the algorithm $\mathcal{R}(x)$ with random bits y, and bet half of the capital on the value predicted by $\mathcal{R}$

For almost every n, this martingale gains by a factor 1.5 at r_n.

Question

PP denotes probabilistic polynomial time: w is in the language if the majority of computations of a polynomial time NTM on input w is accepting. Clearly $\mathsf{PP} \subseteq \mathsf{PSPACE}$.

Is the assumption $\mathsf{P} = \mathsf{PP}$ sufficient for closure of polynomial time randomness under permutations S such that $S, S^{-1} \in \mathsf{P}$?