

Topological isomorphism for classes of closed subgroups of S_∞

André Nies



THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

joint work with Alexander Kechris, Philipp Schlicht and Katrin Tent
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The setting

- ▶ S_∞ is the topological group of permutations of \mathbb{N} .
- ▶ \mathcal{C} is a Borel class of closed subgroups of S_∞ .

We study the complexity of the isomorphism problem for \mathcal{C} :

Given groups G, H in \mathcal{C} ,

how hard is it to determine whether $G \cong H$?

All isomorphisms of groups will be **topological** isomorphisms.

Two antipodal classes

We focus on two classes:

▶ **Oligomorphic:**

for each k , the action on \mathbb{N}^k has only finitely many orbits

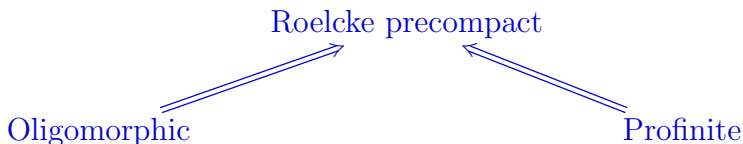
These are the automorphism groups of ω -categorical structures with domain \mathbb{N} .

▶ **Profinite:** each orbit of the action on \mathbb{N} is finite.

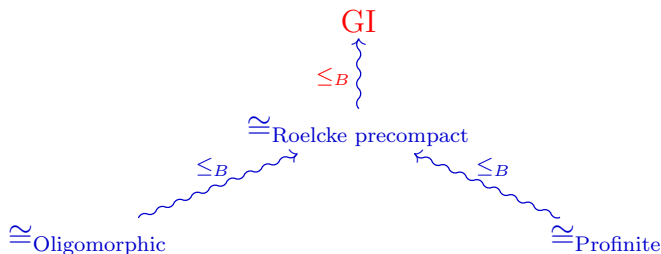
These are up to isomorphism the Galois groups of Galois extensions of countable fields.

A Borel superclass

A closed subgroup G of S_∞ is **Roelcke precompact** if each open subgroup U of G is large in the sense that there is finite set $F \subseteq G$ such that $UFU = G$.



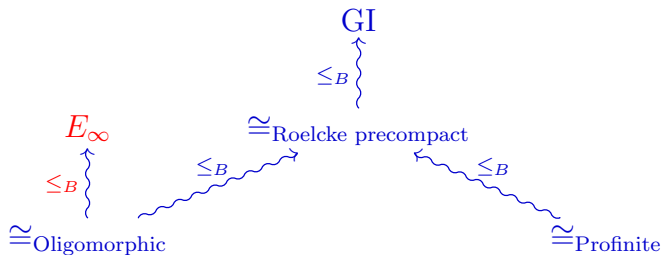
First result (Kechris, N., Tent)



Isomorphism of Roelcke precompact groups is Borel below **GI**.

Graph isomorphism (**GI**) is complete for S_∞ orbit equivalence relations.
Result independently by Rosendal and Zielinski, JSL 2018

Second result (N., Schlicht, Tent)

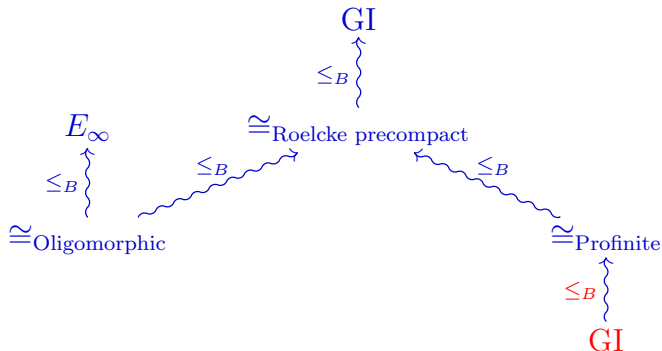


Isomorphism of oligomorphic groups is Borel below E_∞ .

E_∞ denotes a complete countable equivalence relation.

To be countable means: Borel with each class countable.

Third result (Kechris, N., Tent)



Graph isomorphism is Borel below isomorphism of profinite groups.

Since E_∞ is Borel but GI is not, $\cong_{\text{Profinite}} >_B \cong_{\text{Oligomorphic}}$.

The Borel space of closed subgroups of S_∞

For a 1-1 map $\sigma: \{0, \dots, n-1\} \rightarrow \mathbb{N}$ let

$$N_\sigma = \{\alpha \in S_\infty: \sigma \prec \alpha\}$$

The closed subgroups of S_∞ can be seen as points in a standard Borel space. To define the Borel sets, we start with sets of the form

$$\{G \leq_c S_\infty: G \cap N_\sigma \neq \emptyset\},$$

where $G \leq_c S_\infty$ means that G is a closed subgroup of S_∞ .

The Borel sets are generated from these basic sets by complementation and countable union.

Example: for every $\alpha \in S_\infty$, the set $\bigcap_k \{H: H \cap N_{\alpha \upharpoonright k} \neq \emptyset\}$ is Borel. It expresses that a closed subgroup of S_∞ contains α .

Complexity of the isomorphism relation
for Roelcke precompact groups

Roelcke precompactness

A closed subgroup G of S_∞ is called **Roelcke precompact** if for each open subgroup U of G

there is a finite set $F \subseteq G$ such that $UFU = G$.

This condition is **Borel** because it suffices to check it for the basic open subgroups $U_n = \{\rho \in G : \forall i < n [\rho(i) = i]\}$; further, we can pick F from a countable dense set predetermined from G in a Borel way.

FACT. G Roelcke precompact \Rightarrow

G has only countably many open subgroups.

PROOF. Each open subgroup U contains a basic open subgroup U_n . U_n has finitely many double cosets, and U is the union of some of them. In fact, from G we can in a Borel way determine a listing A_0, A_1, \dots without repetition of all open cosets.

Theorem (Kechris, N, Tent, JSL, in press)

Isomorphism of Roelcke precompact groups is Borel reducible to graph isomorphism.

This was independently and via different methods proved by Rosendal and Zielinski (JSL, 2018).

Proof.

- ▶ Let $\mathcal{M}(G)$ be the structure with domain the open cosets. Via the listing A_0, A_1, \dots above, we can identify its domain with ω .
- ▶ The ternary predicate $R(A, B, C)$ holds in $\mathcal{M}(G)$ if $AB \subseteq C$.
- ▶ The main work is to show that for Roelcke precompact $G, H \leq_c S_\infty$,

$$G \cong H \iff \mathcal{M}(G) \cong \mathcal{M}(H).$$

□

Complexity of the isomorphism relation for
oligomorphic subgroups of S_∞

Oligomorphic groups

- ▶ A closed subgroup G of S_∞ is called **oligomorphic** if for each k , the action of G on \mathbb{N}^k has only finitely many orbits.
- ▶ For instance, $\text{Aut}(R)$ and $\text{Aut}(\mathbb{Q}, <)$ are oligomorphic.
- ▶ This is the exact opposite of profinite, where each orbit is finite.
- ▶ Intuitively, oligomorphic groups are big, profinite groups are small.
- ▶ Unlike for profinite groups, G being oligomorphic depends on the way G is embedded into S_∞ .

Oligomorphic groups as automorphism groups

Fact

$G \leq_c S_\infty$ is oligomorphic $\iff G$ is the automorphism group of an ω -categorical structure S with domain \mathbb{N} .

Proof.

\Leftarrow : this follows from the Ryll-Nardzewski Theorem.

\Rightarrow : $G = \text{Aut}(S)$ where S is the structure with a k -ary relation symbol for each orbit of G on \mathbb{N}^k . □

Conjugacy of oligomorphic groups

The conjugacy relation for oligomorphic groups is smooth.

To see this,

- ▶ Given a closed subgroup G of S_∞ , let V_G be the corresponding orbit equivalence structure: for each $k > 0$ introduce a $2k$ -ary relation that holds for two k -tuples of distinct elements if they are in the same orbit of \mathbb{N}^k .
- ▶ V_G is ω -categorical.
- ▶ One checks that for oligomorphic groups G, H

$$G \text{ and } H \text{ are conjugate in } S_\infty \iff V_G \cong V_H.$$

- ▶ Isomorphism of ω -categorical structures M, N for the same language is smooth because $M \cong N \iff \text{Th}(M) = \text{Th}(N)$.

Bi-interpretability

Structures A, B are **bi-interpretable** if there are first-order interpretations Γ, Δ such that

- ▶ $A \cong \Gamma(B)$, $B \cong \Delta(A)$
- ▶ some isomorphism $\gamma : A \cong \widehat{A} = \Gamma(\Delta(A))$ is definable in A , and similarly for $B \cong \Delta(\Gamma(B))$.

(Note that \widehat{A} consists of equivalence classes of tuples from A .)
Coquand¹ showed that for ω -categorical A, B we have

$$\text{Aut}(A) \cong \text{Aut}(B) \iff A \text{ and } B \text{ are bi-interpretable.}$$

¹see Ahlbrandt/Ziegler 1986, or Evans' 2013 Hausdorff Institute notes

The space of theories

- ▶ Theories in a countable language can be seen as infinite bit sequences with an entry for each sentence whether it's in or out. Hence the set of theories inherits a topology from $\{0, 1\}^{\mathbb{N}}$.
- ▶ The complete theories form a closed set.
- ▶ To be ω -categorical is a Π_3^0 property of theories, because by Ryll-Nardzewski this property is equivalent to saying that for each n , the Boolean algebra of formulas with at most n free variables modulo T -equivalence is finite.

Bi-interpretability of structures via their theories

We can express bi-interpretability of ω -categorical structures A, B in terms of their theories:

- ▶ $A \cong \Gamma(B)$ means that $\text{Th}(B)$ says
“the structure interpreted in B via Γ satisfies $\text{Th}(A)$ ”
- ▶ similar for $B \cong \Delta(A)$
- ▶ also express that some $\gamma : A \cong \Gamma(\Delta(A))$ is defined by a particular first order formula.

Bi-interpretability of ω -categorical theories

Theorem (N., Schlicht and Tent)

There is a Σ_2^0 relation which coincides with bi-interpretability on the Π_3^0 set of ω -categorical theories.

Given ω -categorical theories S, T . We have an initial block of existential quantifiers fixing the dimensions of the interpretations and asserting the existence of the definable isomorphism γ .

- ▶ The rest is easy if the signature is finite
- ▶ In general, we have to express that a certain tree computed from S, T is infinite. The tree is matching types of S and types of T in a way consistent with γ being an isomorphism.
- ▶ The branching of the tree is bounded depending on S, T , because the dimensions are fixed, and for each arity there are only so many types. So it is Π_1^0 in S, T to say that the tree is infinite.

Bi-interpretability of ω -categorical theories

Corollary

Bi-interpretability on the set of ω -categorical theories is Borel bi-reducible with a Σ_2^0 -equivalence relation on a Polish space.

Proof of Corollary.

- ▶ There is a finer Polish topology with the same Borel sets in which the set of ω -categorical theories is closed.
- ▶ Then the Σ_2^0 relation above yields a Σ_2^0 description of bi-interpretability on this closed set.



Oligomorphic groups “are” countable models

Theorem

Isomorphism of oligomorphic groups is Borel bi-reducible with the orbit equivalence relation of a Borel action $S_\infty \curvearrowright \mathcal{B}$; where

- ▶ \mathcal{B} is an invariant Borel set of models with domain \mathbb{N} for the language with one ternary relation symbol,
- ▶ the action of S_∞ is the natural one.

A Borel equivalence relation on a Polish space is called **countable** if every equivalence class is countable.

Corollary

Isomorphism of oligomorphic groups is Borel reducible to a countable Borel equivalence relation.

Proof.

- ▶ Above we proved that isomorphism of oligomorphic groups is Borel reducible to a Σ_2^0 equivalence relation on a Polish space.
- ▶ So the isomorphism relation on \mathcal{B} in the foregoing Theorem is Borel reducible to a Σ_2^0 equivalence relation.
- ▶ By Hjorth and Kechris (1995; Theorem 3.8): If an S_∞ orbit equivalence relation is Borel reducible to a Σ_2^0 equivalence relation, then it is reducible to a countable equivalence relation.

Theorem (to discuss)

Isomorphism of oligomorphic groups is Borel bi-reducible with the orbit equivalence relation of the natural action of S_∞ on an isomorphism invariant Borel set \mathcal{B} of models.

For Roelcke precompact G , we defined a structure $\mathcal{M}(G)$ with domain consisting of the cosets of open subgroups. We can in a Borel way find a bijection of these cosets with \mathbb{N} . We showed

$$G \cong H \iff \mathcal{M}(G) \cong \mathcal{M}(H).$$

We will define an “inverse” operation \mathcal{G} of the operation \mathcal{M} on a Borel set \mathcal{B} of models. For oligomorphic G and $M \in \mathcal{B}$ we will have

$$\mathcal{G}(\mathcal{M}(G)) \cong G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M$$

This suffices because it implies the converse reduction

$$\mathcal{G}(M) \cong \mathcal{G}(N) \iff M \cong N.$$

Axiomatizing the range of the map \mathcal{M}

- ▶ We actually define the map \mathcal{G} on an invariant co-analytic set \mathcal{D} of L -structures that contains $\text{range}(\mathcal{M})$.
- ▶ Then $\text{range}(\mathcal{M}) \subseteq \mathcal{B} \subseteq \mathcal{D}$ for an invariant Borel set \mathcal{B} .
- ▶ Since $\mathcal{M}(\mathcal{G}(M)) \cong M$ for each $M \in \mathcal{B}$, actually \mathcal{B} equals the closure of $\text{range}(\mathcal{M})$ under isomorphism.
- ▶ We will observe a number of properties, called **axioms**, of all the structures of the form $\mathcal{M}(G)$. They can be expressed in Π_1^1 form.
- ▶ \mathcal{D} is the set of structures satisfying all the axioms.

Definable relations in $\mathcal{M}(G)$

Recall that our language L only has one ternary relation $R(A, B, C)$ (which is interpreted by $AB \subseteq C$ for cosets A, B, C).

- ▶ The property of A to be a **subgroup** is definable in $\mathcal{M}(G)$ by the formula $AA \subseteq A$. That a subgroup A is contained in a subgroup B is definable by the formula $AB \subseteq B$.
- ▶ A is a **left coset** of a subgroup U if and only if U is the maximum subgroup with $AU \subseteq A$; similarly for A being a **right coset** of U .
- ▶ $A \subseteq B \iff AU \subseteq B$ in case A is a left coset of U .

The first few axioms posit for a general L -structure M that the formulas behave reasonably. E.g., \subseteq is transitive. We use terms like “*subgroup*”, “*left coset of*” to refer to elements satisfying them.

The filter group $\mathcal{F}(M)$

Given a structure M , denote by $\mathcal{F}(M)$ the set of filters (for \subseteq) that contain a left and a right coset of each subgroup. (These cosets are unique because axioms require that distinct left cosets are disjoint etc.) We use letters x, y, z for filters.

$A \in x$ means intuitively that A is an open neighbourhood of the group element x .

With this intuition in mind we define an operation on $\mathcal{F}(M)$:

$$x \cdot y = \{C \in M \mid \exists A \in x \exists B \in y \ AB \subseteq C\}.$$

For A a *right coset* of V and B a *left coset* of V , let $A^* = B$ if $AB \subseteq V$. Let $x^{-1} = \{A^* : A \in x\}$.

The filter of *subgroups* is the identity 1 .

The filter group $\mathcal{F}(M)$

We can express by Π_1^1 axioms that these operations behave as a group: associativity, and $\forall x [x \cdot x^{-1} = 1]$.

The sets $\{x : U \in x\}$, where $U \in M$ is a *subgroup*, are declared a basis of neighbourhoods for the identity. Positing the right axioms, we ensure that $\mathcal{F}(M)$ is a Polish group.

Further, for each subgroup $V \in M$, there is an action $\mathcal{F}(M) \curvearrowright LC(V)$ given by

$$x \cdot A = B \text{ iff } \exists S \in x [SA \subseteq B],$$

where $LC(V)$ denotes the set of *left cosets* of V .

A faithful subgroup

- ▶ For oligomorphic G , there is an open subgroup V such that the action $G \curvearrowright LC(V)$ is oligomorphic:
e.g. let $V = G_{\{n_1, \dots, n_k\}}$ where the n_i represent the k many 1-orbits. Call such a V a **faithful** subgroup.
- ▶ As a further axiom for an abstract L -structure M , we require the existence of such V , and that the embedding of $\mathcal{F}(M)$ into S_∞ is topological (these axioms are Π_1^1 but not first-order).
- ▶ Then $\mathcal{F}(M)$ is oligomorphic and hence Roelcke precompact.

Showing that the coset structure of $\mathcal{F}(M)$ is isomorphic to M

Mainly, we have to show that each open subgroup \mathcal{U} of $\mathcal{F}(M)$ has the form $\mathcal{U} = \{x : U \in x\}$ for some *subgroup* U in M .

- ▶ By definition of the topology, \mathcal{U} contains a basic open subgroup $\widehat{W} = \{x : W \in x\}$, for some *subgroup* $W \in M$.
- ▶ Since $\mathcal{F}(M)$ is Roelcke precompact, \mathcal{U} is a finite union of double cosets of \widehat{W} .
- ▶ We require as an axiom for M that each such finite union that is closed under the group operations corresponds to an actual *subgroup* in M .

Turning $\mathcal{F}(M)$ into closed subgroup $\mathcal{G}(M)$ of S_∞

- ▶ By Π_1^1 uniformization (Addison/Kondo), from $M \in \mathcal{B}$ we can in a Borel way determine a faithful subgroup V .
- ▶ Let A_0, A_1, \dots list $LC(V)$ in the natural order.
- ▶ Then the action $\mathcal{F}(M) \curvearrowright LC(V)$ yields a topological embedding of $\mathcal{F}(M)$ into S_∞ .
- ▶ Its range is the desired closed subgroup $\mathcal{G}(M)$.

By the arguments above we have for each oligomorphic G and each $M \in \mathcal{B}$

$$\mathcal{G}(\mathcal{M}(G)) \cong G \text{ and } \mathcal{M}(\mathcal{G}(M)) \cong M.$$

Theorem (Finished)

Isomorphism of oligomorphic groups is Borel bi-reducible with the orbit equivalence relation of the natural action of S_∞ on a Borel set \mathcal{B} of models.

Complexity of the isomorphism relation between
profinite subgroups of S_∞

Definition

A topological group G is called **profinite** if one of the following equivalent conditions holds.

- (a) G is compact, and the clopen sets form a basis for the topology (i.e., G is totally disconnected).
- (b) G is the inverse limit of a system of finite groups carrying the discrete topology.

We assume all topological groups are separable. In this case, we have a further equivalent condition

- (c) G is isomorphic to a closed subgroup of S_∞ with all orbits finite.

Topologically finitely generated profinite groups

The isomorphism relation $E_{f.g.}$ between finitely generated profinite groups G is Borel bi-reducible with $\text{id}_{\mathbb{R}}$.

There are uncountably many non-isomorphic such G . So by Silver's theorem it suffices to show $E_{f.g.} \leq_B \text{id}_{\mathbb{R}}$.

- ▶ A finitely generated profinite group G is determined by its isomorphism types of finite quotients.
- ▶ Let $q(G)$ be the set of these isomorphism types, written in some fixed way as an infinite bit sequence. This map is Borel because from G one can “determine” its finite quotients².
- ▶ Then $G \cong H \iff q(G) = q(H)$. So $E_{f.g.}$ is smooth.

²e.g. Fried/Jarden, Field arithmetic, 16.10.7

Complexity of isomorphism for Abelian profinite groups

Pontryagin duality associates to each abelian locally compact group G the group G^* of continuous homomorphisms from G into the circle \mathbb{T} , with the compact-open topology.

Pontryagin duality theorem: $G \cong (G^*)^*$ via the application map $g \mapsto \lambda\phi. \phi(g)$, for each locally compact abelian group G .

Countable abelian torsion versus abelian profinite

A special case of this states that (discrete) countable abelian torsion groups A correspond to abelian profinite groups. We have $A \cong B$ iff $A^* \cong B^*$. The functor $*$ and its inverse are Borel.

So isomorphism between countable abelian torsion groups is Borel equivalent to continuous isomorphism between abelian profinite groups.

Isomorphism between abelian countable torsion groups is neither smooth, nor above graph isomorphism (H. Friedman and Stanley, 1989).

Graph isomorphism \leq_B isom. of profinite groups

A group G is nilpotent-2 if it satisfies the law $[[x, y], z] = 1$.

Let \mathcal{N}_2^p denote the variety of nilpotent-2 groups of exponent p .

Theorem

Let $p \geq 3$ be prime. Graph isomorphism can be Borel reduced to isomorphism between profinite groups in \mathcal{N}_2^p .

Proof: A result of Alan Mekler (1981) implies the theorem for countable abstract groups. We adapt his construction to the profinite setting.

A symmetric and irreflexive countable graph is called **nice** if it has no triangles, no squares, and for each pair of distinct vertices x, y , there is a vertex z joined to x and not to y .

Easy fact: Graph isomorphism \leq_B nice graph isomorphism.

Mekler's construction

isomorphism of nice graphs \leq_B

isomorphism of countable groups in \mathcal{N}_2^p .

▶ Let F be the free \mathcal{N}_2^p group on free generators x_0, x_1, \dots

▶ For $r \neq s$ we write $x_{r,s} = [x_r, x_s]$.

▶ Given a graph with domain \mathbb{N} and edge relation A , let

$$G(A) = F / \langle x_{r,s} : rAs \rangle_{\text{normal closure}}.$$

▶ The centre of $G(A)$ is abelian of exponent p with a basis consisting of the $x_{r,s}$ such that $\neg rAs$.

Show that A can be reconstructed from $G(A)$. Therefore:

Let A, B be a nice graphs. Then $A \cong B$ iff $G(A) \cong G(B)$.

Topological version of Mekler's construction

- ▶ Elements of $G(A) = F/\langle x_{r,s} : rAs \rangle$ have unique normal form

$$\prod_{\langle r,s \rangle \in L} x_{r,s}^{\beta_{rs}} \prod_{i \in D} x_i^{\alpha_i}, \quad 0 < \alpha_i, \beta_{rs} < p,$$

where L is a finite set of non-edges, D a finite set of vertices.

- ▶ Let R_n be the normal subgroup of $G(A)$ generated by the x_i , $i \geq n$. Let $\overline{G}(A)$ be the completion of $G(A)$ w.r.t. the R_n , i.e.,

$$\overline{G}(A) = \varprojlim_n G(A)/R_n.$$

- ▶ Each $G(A)/R_n$ is finite, so this is a profinite group.
- ▶ Elements of $\overline{G}(A)$ have a normal form $\prod_{\langle r,s \rangle \in L} x_{r,s}^{\beta_{rs}} \prod_{i \in D} x_i^{\alpha_i}$, where L and D are now allowed to be infinite.

One verifies that A can be reconstructed from $\overline{G}(A)$:

Let A, B be a nice graphs. Then $A \cong B$ iff $\overline{G}(A) \cong \overline{G}(B)$.

$A \rightarrow \overline{G}(A)$ is Borel. So $\mathbf{GI} \leq_B$ isomorphism of profinite \mathcal{N}_2^p groups.

A fake reduction from $\cong_{\text{Profinite}}$ to $\cong_{\text{Oligomorphic}}$

- ▶ Evans and Hewitt (1990): every profinite group is a topological quotient of an oligomorphic group.
- ▶ Given oligomorphic G , let $D(G)$ be the intersection of all open subgroups with finite index. Note that $D(G)$ is closed and invariant.
- ▶ They show that up to isomorphism, $G/D(G)$ ranges through all separable profinite groups P .

However, from profinite $P \leq_c S_\infty$, we **cannot** Borel determine oligomorphic G_P such that $G_P/D(G_P) \cong P$ in such a way that $P \cong Q$ implies $G_P \cong G_Q$. In their construction, one can see how G_P depends on the way P is presented as a subgroup of S_∞ .

Some open problems

- ▶ How complex is isomorphism of arbitrary closed subgroups of S_∞ ? Is it \leq_B -complete for analytic equivalence relations?
- ▶ What is a lower bound for the complexity of isomorphism for oligomorphic groups?
- ▶ How about if the signature of a corresponding ω -categorical structure can be made finite?

References

- ▶ Kechris, N. and Tent,
The complexity of topological group isomorphism,
JSL, in press, arXiv: 1705.08081
- ▶ N., Schlicht and Tent, The complexity of oligomorphic group isomorphism, in preparation.