Structure within the class of K-trivial sets

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Plan

- Brief introduction to randomness
  and its connection to other fields
- Anti-randomness (or K-triviality) coincides with
  being close to computable
- Absolute and relative complexity within the class of K-trivials
  - defined via subclasses,
  - defined via the new "ML-reducibility",

  respectively.

Martin-Löf randomness

Our central algorithmic randomness notion is the one of Martin-Löf. It has many equivalent definitions. Here is one.

Z is Martin-Löf random $\iff$

> for every computable sequence $(\sigma_i)_{i\in\mathbb{N}}$ of binary strings with $\sum_i 2^{-|\sigma_i|} < \infty$, there are only finitely many i such that σ_i is an initial segment of Z.

Note that $\lim_i 2^{-|\sigma_i|} = 0$, so this means that we cannot "Vitali cover" Z, viewed as a real number, with the collection of dyadic intervals corresponding to $(\sigma_i)_{i\in\mathbb{N}}$.

Definition via ML-tests

Sets are viewed as points in Cantor space $\{0,1\}^{\mathbb{N}}$.
Let λ denote the uniform (product) measure on $\{0,1\}^{\mathbb{N}}$.

- A ML-test is a uniformly computably enumerable sequence $(G_m)_{m\in\mathbb{N}}$ of open sets in $\{0,1\}^{\mathbb{N}}$ such that $\lambda G_m \leq 2^{-m}$ for each m.
- A set Z is ML-random if Z passes each ML-test, in the sense that Z is not in all of the G_m.

There is a universal ML-test (S_r): a set Z is ML-random iff it passes (S_r).

Characterise ML-randomness via differentiability

Theorem (Demuth 1975/Brattka, Miller, Nies 2011)

Let $r \in [0, 1]$. *Then*

r *is ML-random* $\iff$

$f'(r)$ exists, for each function f of bounded variation such that $f(q)$ is a computable real, uniformly in each rational q.

- The implication "$\Rightarrow$" is an effective version of the classical theorem.
- The implication "$\Leftarrow$" has no classical counterpart. To prove it, one builds a computable function f of bounded variation that is only differentiable at ML-random reals.

Density

Let λ denote uniform (Lebesgue) measure.

Definition

Let E be a measurable subset of $[0,1]$. The (lower) density of E at a real z is

$$\underline{\rho}(E \mid z) = \liminf_{z \in J\,,\,|J| \to 0} \frac{\lambda(J \cap E)}{|J|},$$

where J ranges over intervals.

This gauges how much, at least, of E is in intervals that zoom in on z.

$\rho(E \mid z)$ is the limit over intervals containing z. Clearly

$$\underline{\rho}(E \mid z) = 1 \leftrightarrow \rho(E \mid z) = 1.$$

Lebesgue's Density Theorem

Theorem (Lebesgue Density Theorem, 1910)

Let $E \subseteq [0,1]$ be measurable. Then for almost every $z \in [0,1]$:

if $z \in E$, then $\underline{\rho}(E \mid z) = 1$.

It is sufficient to prove it for closed E, because closed sets approximate E from inside.

Martin-Löf randomness and density

Does Martin-Löf randomness ensure that an effectively closed $E \subseteq [0,1]$ with $z \in E$ has density one at z?

Answer: NO!

Example

- Let $E \neq \emptyset$, $E \subseteq [0,1]$ be an effectively closed set containing only Martin-Löf randoms.
- E.g., $E = [0,1] \setminus S_1$ where $\langle S_r \rangle_{r \in \mathbb{N}}$ is a universal ML-test.
- Let $z = \min(E)$.
- Then $\underline{\rho}(E \mid z) = 0$ even though z is ML-random.

(This uses that every ML-random is Borel normal.)

Density randomness

A ML-random real z is density random if the conclusion of Lebesgue's theorem holds for each effectively closed E containing z.

- Cantor space version (and dyadic density) is equivalent
- This notion is equivalent to left-r.e. martingale convergence, and differentiability of "interval-r.e." functions (Madison group 2012; Myabe, Nies and Zhang BSL 2016)
- Day and Miller (2015) built a ML-random $z \not\geq_T \emptyset'$ which is not density random. So Turing incompleteness of a ML-random real is not sufficient for density randomness.
- I.o.w. density random properly implies "difference random" (which is equivalent to positive density).

Anti-random, or K-trivial sets

Definition of K-triviality

$K(x)$ is the prefix-free complexity of string x.

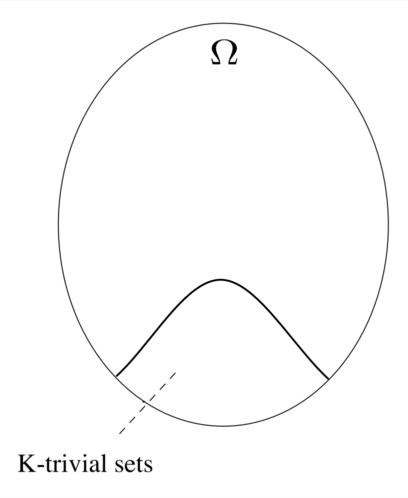
Definition (going back to Chaitin, 1975)

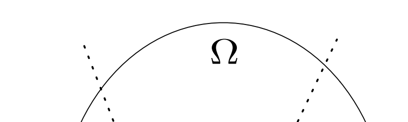
An infinite sequence of bits A is K-trivial if, for some $b \in \mathbb{N}$,

$$\forall n \, [K(A \upharpoonright_n) \leq K(0^n) + b],$$

namely, all its initial segments have minimal K-complexity.

Some properties of the K-trivials





Far from random $=$ close to computable

- An oracle $A \subseteq \mathbb{N}$ is low for Martin-Löf randomness if every random set is already random with oracle A.
- That is, A cannot "derandomize" any random set.
- This means that A is very close to computable.

The following says that far from random $=$ close to computable.

Theorem (N., 2005)

Let $A \subseteq \mathbb{N}$*. Then*

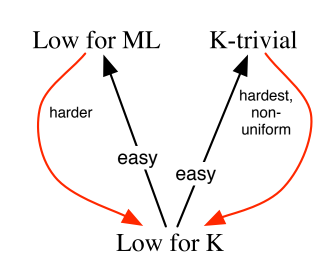
$$A \textit{ is } K\textit{-trivial} \iff A \textit{ is low for Martin-Löf randomness.}$$

Lowness for K

A is called low for K (Muchnik, 1998) if enhancing the computational power of the universal function $\mathbb{U}$ by an oracle A does not decrease $K(y)$:

$$\forall y \; [K(y) \leq K^A(y) + O(1)].$$

- The straightforward implications are
  low for $K \Rightarrow$ low for ML and
  low for $K \Rightarrow K$-trivial.
- In N (2005) the converse implications are shown.



A more recent equivalence for K-trivial c.e. sets

Theorem (BGKNT 16 + Day, Miller 16)

Let $A \subseteq \mathbb{N}$ be c.e.

$$A \text{ is } K\text{-trivial} \iff A \leq_T Z \text{ for some ML-random set } Z \not\geq_T \emptyset'.$$

Proof idea:

- If Z is ML-random and fails the conclusion of the Lebesgue density theorem, then Z computes all the K-trivials.
- By Day and Miller's theorem there is such a Z that is Turing incomplete (and even Δ^0_2).

So in fact a single ML-random $Z <_T \emptyset'$ suffices to Turing-cover all the K-trivials.

This also yields a new proof of K-trivial $\Rightarrow$ low for K.

Dynamic characterisation of the K-trivials

Definition of cost functions

Definition

A cost function is a computable function

$$c : \mathbb{N} \times \mathbb{N} \to \{x \in \mathbb{Q} \colon\ x \geq 0\}.$$

We say that c is monotonic if $c(x, s)$ is nonincreasing in x, and nondecreasing in s.

When building a computable approximation of a Δ^0_2 set A, we view $c(x, s)$ as the cost of changing $A(x)$ at stage s.

Obeying a cost function

We want to make the total cost of changes, taken over all x, finite.

Definition

The computable approximation $(A_s)_{s\in\mathbb{N}}$ obeys a cost function c if

$$\infty > \sum_{x,s} c(x,s) [\![x < s \wedge x \text{ is least s.t. } A_{s-1}(x) \neq A_s(x)]\!].$$

For a Δ^0_2 set A, we write $A \models c$ (A obeys c) if some computable approximation of A obeys c.

Basic existence theorem

We say that a cost function c satisfies the limit condition if

$$\lim_x \sup_s c(x,s) = 0.$$

Theorem (Kučera, Terwijn 1999; D,H,N,S 2003; ...)

If a cost function c satisfies the limit condition, then some promptly simple set A obeys c.

Theorem (N. to appear in this version)

A is K-trivial $\iff$ A obeys the cost function $c_\Omega(x,s) = \Omega_s - \Omega_x$.

Here Ω is Chaitin's number. One can in fact take any left-c.e. ML-random real.

Subideals of the K-trivials

Even though K-trivials are very close to computable, a lot of structure has emerged inside that class.

Title of a talk of R. Feynman: *"There's lots of space down there"*

Below both halves of a ML-random

The "halves" of a set $Z \subseteq \mathbb{N}$ are

$$Z_0 = \{n\colon 2n \in Z\} \text{ and } Z_1 = \{n\colon 2n+1 \in Z\}.$$

Theorem (Greenberg, Miller, N., submitted)

The following are equivalent for a Δ^0_2 set A.

- *A is Turing below both halves of a ML-random set*
- *A is Turing below both halves of Ω*
- *A obeys the cost function $c(x,s) = \sqrt{\Omega_s - \Omega_x}$.*

Obeying a cost function is closed under $\oplus$. So the first and third condition together show that the sets of this kind form a Turing ideal.

Below p out of q columns of a ML-random

The q-columns of a set $Z \subseteq \mathbb{N}$ are $Z_p = \{n \colon qn + p \in Z\}$ for $p < q$.

Theorem (Greenberg, Miller, N., ibd.)

The following are equivalent for a Δ^0_2 set A.

- *A is Turing below the disjoint sum of every p out of q columns of a ML-random set Z*
- *Same for $Z = \Omega$*
- *A obeys the cost function $c(x, s) = (\Omega_s - \Omega_x)^{p/q}$.*

Again, sets of this kind form an ideal. Call it $\mathcal{B}_{p/q}$. By general facts on cost functions (N. ta), $\alpha < \beta \Rightarrow \mathcal{B}_\alpha \subset \mathcal{B}_\beta$.

We also showed that $\bigcup_{\alpha<1} \mathcal{B}_\alpha$ is the class of sets that are robustly computable from some ML-random Z (i.e. computable from all coarse descriptions of Z).

Relative complexity of K-trivials

ML-reducibility

Some foregoing results suggest that the complexity of a K-trivial is largely determined by its interaction with ML-random sets. For sets $A, B \subseteq \mathbb{N}$ we define

$$A \leq_{\mathrm{ML}} B \iff \forall Z \in \mathsf{MLR}\,[B \leq_{\mathrm{T}} Z \to A \leq_{\mathrm{T}} Z].$$

This only measure the aspects of complexity of a K-trivial "known" to ML-random oracles. Clearly $\leq_{\mathrm{T}}$ implies $\leq_{ML}$, and the least degree is the computable sets.

Theorem (restrict to ML-degrees of c.e. K-trivials)

- *The K-trivials are closed downward under $\leq_{\mathrm{ML}}$.*
- *If B is c.e. and NOT K-trivial, then any MLR $\geq_T B$ computes $\emptyset'$, so $A \leq_{\mathrm{ML}} B$ for each Δ^0_2 set A.*
- *(GMNT, in prep.) For all A K-trivial there is a c.e. $D \geq_{tt} A$ such that $D \leq_{\mathrm{ML}} A$. In particular D is K-trivial.*

$\mathbf{c}$-tests and smartness for a cost function $\mathbf{c}$

- For a cost function $\mathbf{c}$ with the limit condition write $\underline{\mathbf{c}}(m) = \lim_s \mathbf{c}(m, s)$.
- Suppose $\mathbf{c}$ is a cost function with $\mathbf{c}(m) \geq 2^{-m}$. A uniformly Σ^0_1 sequence (G_m) is a $\mathbf{c}$-test if $\lambda G_m \leq \mathbf{c}(m)$.
- A is smart for $\mathbf{c}$ if $A \models \mathbf{c}$ and no $Y \geq_T A$ is $\mathbf{c}$-random.

Theorem (BGKNT, JEMS 2016)

Some c.e. set A is smart for $\mathbf{c}_\Omega$.

Smart sets exist for cost functions

Theorem (GMNT, in prep)

For each $\mathbf{c} \geq \mathbf{c}_\Omega$, *some c.e.* A *is smart for* $\mathbf{c}$.

So A is smart for $\mathbf{c}$ iff A is $\leq_{\mathrm{ML}}$-complete among the sets obeying $\mathbf{c}$.

- E.g., for positive rational $p/q = \alpha < 1$ let A_α be smart for $\mathbf{c}_\alpha(x, s) = (\Omega_s - \Omega_x)^\alpha$.
- Recall that $\mathcal{B}_{p/q}$ is the class of A that Turing below the disjoint sum of every p out of q columns of a ML-random set Z.
- We have $\mathcal{B}_\alpha = \{B\colon\ B \leq_{\mathrm{ML}} A_\alpha\}$.
  $\subseteq$ by the theorem, $\supseteq$ because $\mathcal{B}_\alpha$ is downward closed under $\leq_{\mathrm{ML}}$.
- The ML-degrees of the A_α form a dense linear order.

Structure of the ML-degrees of K-trivials

Theorem (Kučera, essentially)

For each c.e., incomputable D there are $A, B \leq_{\mathrm{T}} D$ such that $A \mid_{ML} B$.

Theorem (GMNT, in prep)

There is no ML-minimal pair.

Proof idea: For each c.e. K-trivial A we have a cost function $\mathbf{c}_A$ with $A \models \mathbf{c}_A$ such that for MLR Y,

$$Y \geq_T A \iff Y \text{ is not } \mathbf{c}_A\text{-random.}$$

Let $\mathbf{c} = \mathbf{c}_A + \mathbf{c}_B$ and take simple $D \models \mathbf{c}$. Then $D \leq_{\mathrm{ML}} A, B$.

Questions

- Is $<_{ML}$ dense on the K-trivials?
- Is being a smart K-trival an arithmetical property? Stronger: is $\leq_{\mathrm{ML}}$ an arithmetical relation?
- Can a smart K-trivial be cappable?
- Does every nonzero ML-degree contain a Turing minimal pair?

A draft of this work is available on the 2016 Logic Blog.