Sets that coincide with $1/2 - \epsilon$ of each computable set

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The Γ parameter of a Turing degree

For $Z \subseteq \mathbb{N}$ the lower density is defined to be

$$\underline{\rho}(Z) = \liminf_n \frac{|Z \cap [0,n)|}{n}.$$

Recall that

$$\gamma(A) = \sup_{X \text{ computable}} \underline{\rho}(A \leftrightarrow X)$$

The Γ parameter was introduced by Andrews et al. (2013):

$$\Gamma(A) = \inf\{\gamma(Y) \colon\ Y \leq_T A\}.$$

Theorem (Monin, 2016, available on Logic Blog 2016)

$\Gamma(A)$ *is either* 0, *or* $1/2$, *or* 1. *Also* $\Gamma(A) = 0 \Leftrightarrow \exists f \leq_{\mathrm{T}} A$ $\forall g$ *computable, bounded by* $2^{(2^n)} \exists^\infty n\ f(n) = g(n)]$

Viewing $1 - \Gamma$ as a Hausdorff pseudodistance

For $Z \subseteq \mathbb{N}$ the upper density is defined by

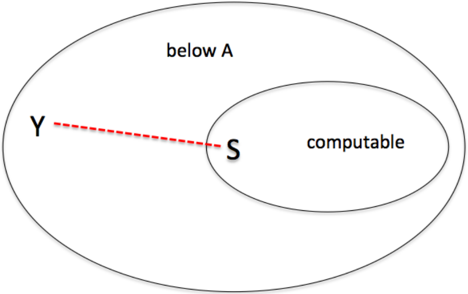
$$\overline{\rho}(Z) = \limsup_n \frac{|Z \cap [0, n]|}{n}.$$

- For $X, Y \in 2^{\mathbb{N}}$ let $d(X, Y) = \overline{\rho}(X \triangle Y)$ be the upper density of the symmetric difference of X and Y
- this is a pseudodistance on Cantor space $2^{\mathbb{N}}$ (that is, two objects may have distance 0 without being equal).

Let $\mathcal{R} \subseteq \mathcal{A} \subseteq M$ for a pseudometric space(M, d). The Hausdorff distance is $d_H(\mathcal{A}, \mathcal{R}) = \sup_{Y \in \mathcal{A}} \inf_{S \in \mathcal{R}} d(Y, S))$.

Given an oracle set A let $\mathcal{A} = \{Y : Y \leq_{\mathrm{T}} A\}$. Let $\mathcal{R} \subseteq \mathcal{A}$ denote the collection of computable sets. We have

$$1 - \Gamma(A) = d_H(\mathcal{A}, \mathcal{R}).$$



Δ parameter of a Turing degree

$$\begin{aligned} \delta(Y) &= \inf\{\underline{\rho}(Y \leftrightarrow S) \colon S \text{ computable}\} \\ \Delta(A) &= \sup\{\delta(Y) \colon Y \leq_T A\}. \end{aligned}$$

- $\Gamma(A)$ measures how well computable sets can approximate the sets that A computes.
  "$\Gamma(A) > p$" for fixed $p \in [0,1)$ is a lowness property.
- $\Delta(A)$ measures how well the sets that A computes can approximate the computable sets.
  "$\Delta(A) > p$" is a highness property.

$$\begin{aligned}\delta(Y) &= \inf\{\underline{\rho}(Y \leftrightarrow S)\colon\ S \text{ computable}\}\\ \Delta(A) &= \sup\{\delta(Y)\colon\ Y \leq_T A\}.\end{aligned}$$

Properties of δ and Δ (w. Merkle and Stephan, Feb 2016)

- $\delta(Y) \leq 1/2$ for each Y (by considering also the complement of S)
- Y Schnorr random $\Rightarrow \delta(Y) = 1/2$ (by law of large numbers)
- A computable $\Rightarrow \Delta(A) = 0$.
- $\Delta(A) = 0$ is possible for noncomputable A, e.g. if A is low and c.e., or A is 2-generic.

The highness classes $\mathcal{B}(p)$

Definition (Brendle and N.)

For $p \in [0, 1/2)$ let

$$\mathcal{B}(p) = \{A\colon\ \exists Y \leq_{\mathrm{T}} A\, \forall S \text{ computable } \underline{\rho}(Y \leftrightarrow S) > p\}.$$

$$\Delta(A) > p \Rightarrow A \in \mathcal{B}(p) \Rightarrow \Delta(A) \geq p.$$

We will show that all the classes $\mathcal{B}(p)$ coincide, for $0 < p < 1/2$. Therefore:

$$\Delta(A) > 0 \Rightarrow \Delta(A) = 1/2.$$

Almost everywhere avoiding a comp. function

Definition ($\mathcal{B}(\neq^*, h)$, also known as $\mathtt{SNR}_h$)

For a function h, we let
$$\mathcal{B}(\neq^*, h) = \{A\colon\ \exists f \leq_{\mathrm{T}} A, f < h\,\forall g \text{ computable}\ \forall^\infty n\, f(n) \neq g(n)\}.$$

- This gets easier as h grows faster.
- The largest class $\mathcal{B}(\neq^*, \infty)$ coincides with "high or diagonally noncomputable".

  (Kjos-Hanssen, Merkle and Stephan, TAMS, Thm 5.1)
- outside the high sets, the hierarchy is closely related to the hierarchy of computing a DNR function below $\hat{h}$.

Fact

A computes a Schnorr random $\Rightarrow$
$A \in \mathcal{B}(\neq^*, 2^{\hat{h}})$ whenever $\hat{h}$ is computable and $\infty > \sum_n 1/\hat{h}(n)$ is computable. E.g. $\hat{h}(n) = n^2$.

Main result

Recall $\mathcal{B}(p) = \{A\colon\ \exists Y \leq_{\mathrm{T}} A\, \forall S \text{ computable } \underline{\rho}(S \leftrightarrow Y) > p\}$.

$\mathcal{B}(\neq^*, h) = \{A\colon\ \exists f \leq_{\mathrm{T}} A, f < h\, \forall g \text{ computable}$

$\forall^\infty n\, g(n) \neq f(n)\}$.

Theorem (N., dual form of Monin's result)

$\mathcal{B}(p) = \mathcal{B}(\neq^*, 2^{(2^n)})$ for each $p \in (0, 1/2)$.

Corollary

$\Delta(A) > 0 \Leftrightarrow \Delta(A) = 1/2 \Leftrightarrow A \in \mathcal{B}(\neq^*, 2^{(2^n)})$.

Recalling that $B(\neq^*, 2^{(2^n)}) \subseteq B(\neq^*, \infty) =$ high $\vee$ d.n.c:

Corollary

$\Delta(A) > 0 \Rightarrow$ A is high or d.n.c.

View as mass problems

We can also view $\mathcal{B}(p)$ and $\mathcal{B}(\neq^*, h)$ as mass problems (i.e. subsets of ω^ω). Re-define

$\mathcal{B}(p) = \{Y \in 2^{\mathbb{N}} \colon \forall S \text{ computable } \underline{\rho}(S \leftrightarrow Y) > p\}$.

$\mathcal{B}(\neq^*, h) = \{f < h \colon \forall g \text{ computable } \forall^\infty n\, g(n) \neq f(n)\}$.

Let $\leq_S$ denote uniform (or Medvedev) reducibility.
Unlike Monin's result, here we have Medvedev reductions.

Theorem (strengthens previous theorem)

$\mathcal{B}(p) \equiv_S \mathcal{B}(\neq^*, 2^{(2^n)})$ for each $p \in (0, 1/2)$.

Easier direction (1)

Proposition

Let $p \in (0, 1/2)$. We have $\mathcal{B}(p) \geq_S \mathcal{B}(\neq^*, 2^{(2^n)})$.

Pick $a \in \mathbb{N}$ with $2/a < p$.

Claim (1)

$\mathcal{B}(2/a) \geq_S \mathcal{B}(\neq^*, 2^{(a^n)})$.

Proof.

Let (I_n) be the consecutive intervals in $\mathbb{N}^+$ of length a^n. Then $|I_n| > (a-1)|\bigcup_{k<n} I_k|$. So

$$X \in \mathcal{B}(2/a) \Rightarrow \forall U \text{ comp. } \forall^\infty n\, X \upharpoonright I_n \neq U \upharpoonright I_n$$

because $\underline{\rho}(X \leftrightarrow U^c) > 2/a$. The class on the right is Medvedev equivalent to $\mathcal{B}(\neq^*, 2^{(a^n)})$. $\square$

Easier direction (2)

Proposition (recall)

Let $p \in (0, 1/2)$. We have $\mathcal{B}(p) \geq_S \mathcal{B}(\neq^*, 2^{(2^n)})$.

Claim (2)

$\mathcal{B}(\neq^*, h(n)) \equiv_S \mathcal{B}(\neq^*, h(2n))$ for each nondecreasing h.

Proof.

$\geq_S$ is trivial. For $\leq_S$:
Given $f \in \mathcal{B}(\neq^*, h(2n))$, let $g(2n) = g(2n+1) = f(n)$.
Then $g \in \mathcal{B}(\neq^*, h(n))$. $\square$

Iterating this $\log_2 a$ times we get

$$B(\neq^*, 2^{(a^n)}) \equiv_S B(\neq^*, 2^{(2^n)}).$$

Sketch the harder direction $\mathcal{B}(p) \leq_S \mathcal{B}(\neq^*, 2^{(2^n)})$:

Relation 1: Let $q > p$ such that $q < 1/2$. For $h(n) = 2^{\hat{h}(n)}$ and functions $x, y < h$, view $x(n)$ as string of length $\hat{h}(n)$.

$$x \neq^*_{\hat{h},q} y \Leftrightarrow \forall^\infty n \, |\{i < \hat{h}(n) \colon x(n)(i) \neq y(n)(i)\}| \geq \hat{h}(n)q.$$

Relation 2: Let $L \in \mathbb{N}$ and u be a function. For a trace s consisting of L-element sets, and a function $y < u$, let

$$s \not\ni^*_{u,L} y \Leftrightarrow \forall^\infty n[s(n) \not\ni y(n)].$$

Define $\mathcal{B}$-classes for these relations as before. Four steps:

1. there is k such that where $\hat{h}(n) = \lfloor 2^{n/k} \rfloor$
$$\mathcal{B}(p) \leq_S \mathcal{B}(\neq^*_{\hat{h},q}).$$
2. There are $L \in \mathbb{N}$, $\epsilon > 0$ such that where $u(n) = 2^{\lfloor \epsilon \hat{h}(n) \rfloor}$, we have $\mathcal{B}(\neq^*_{\hat{h},q}) \leq_S \mathcal{B}(\not\ni^*_{u,L})$ using error correction.
3. $\mathcal{B}(\not\ni^*_{u,L}) \quad \leq_S \quad \mathcal{B}(\not\ni^*_{2^{(L2^n)},L})$.
4. Finally, $\mathcal{B}(\not\ni^*_{2^{(L2^n)},L}) \leq_S \mathcal{B}(\neq^*, 2^{(2^n)})$

Separations?

By the easy direction above, $\mathcal{B}(0) \geq_S B(\neq^*, 2^{n!})$.

Question

Is $\mathcal{B}(1/4) \equiv_S \mathcal{B}(\neq^*, 2^{2^n}) >_W \mathcal{B}(0)$?

When do we know $\mathcal{B}(\neq^*, g) >_W \mathcal{B}(\neq^*, h)$? E.g.

- $g(n) = 2^{n^2}, h(n) = 2^{2^n}$, or
- $g(n) = 2^{2^n}, h(n) = 2^{n!}$?

Work in progress with Khan and Kjos-Hanssen, building on work of Khan and Miller on forcing with bushy trees:

- (Down) For each order function g there is order function h with $h > g$ such that $\mathcal{B}(\neq^*, g) >_W \mathcal{B}(\neq^*, h)$.
- (Up) For each order function h there is order function g with $h > g$ such that $\mathcal{B}(\neq^*, g) >_W \mathcal{B}(\neq^*, h)$.

References

J. Brendle, A. Brooke-Taylor, Keng Meng Ng, and A. Nies.
An analogy between cardinal characteristics and highness properties of oracles. In *Proceedings of the 13th Asian Logic Conference: Guangzhou, China*, pages 1–28. World Scientific, 2013. `http://arxiv.org/abs/1404.2839`.

B. Monin and A. Nies. A unifying approach to the Gamma question. In *Proceedings of Logic in Computer Science (LICS)*. IEEE press, 2015.

B. Monin. Forthcoming conference paper on answer to the Gamma question

B. Monin, A. Nies. Forthcoming journal paper

A. Nies (editor). Logic Blog 2016.
Available from my home page (on arXiv shortly), 2016.

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