Lowness, Randomness, and Computable Analysis

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Pursuit of the Universal

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The setting of computability theory

The objects of study are infinite sequences of bits, identified with sets of natural numbers.

Computability theory studies their complexity.

Basic distinction: computable, or not.

A lowness notion provides a sense in which a sequence of bits is close to computable.

Two examples of lowness notions

An oracle set *A* is computably dominated if functions that A computes don't grow too fast: every such function f is below a computable function g.

An oracle set A is low if the halting set relative to A has the least possible complexity:

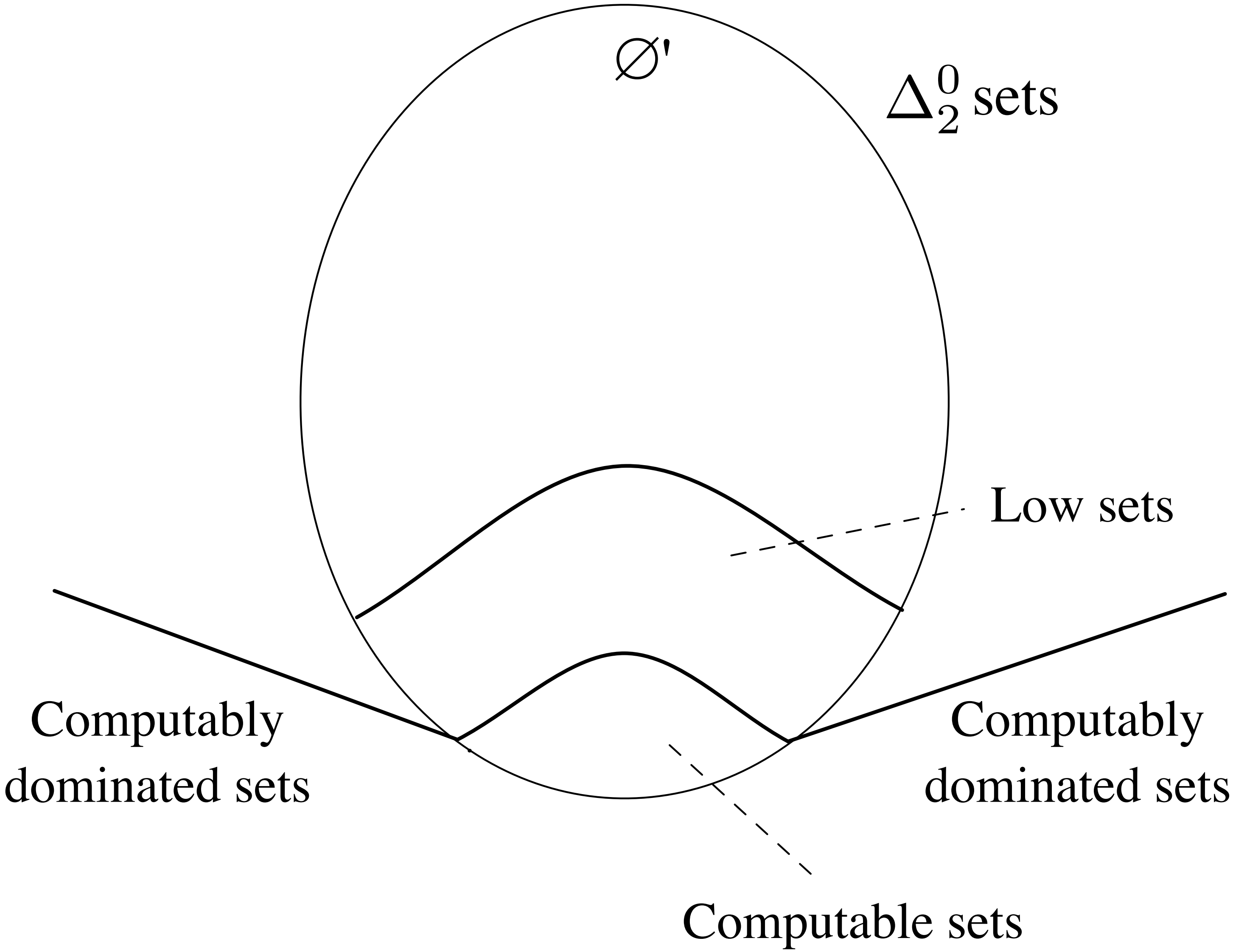
$$A' \equiv_T \emptyset'$$

These lowness notions are incompatible

A computably dominated oracle A cannot be Turing below the halting set, unless A is computable.

So being computably dominated and being low are incompatible outside the computable!

Diagram of two lowness notions

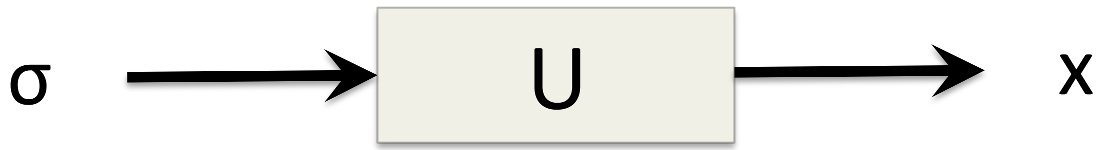


Randomness ←—— Lowness

K-triviality

Kolmogorov complexity C(x)

- For a finite bit sequence *x*, let *C(x)* denote the shortest length of a compressed form of *x* (Solomonoff/Kolmogorov).
- We use a *universal de-compressor U*.
- C(x) is the length of a shortest σ such that U(σ) = x.



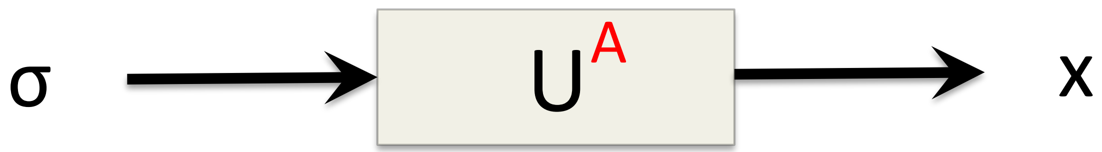
Prefix-free Kolmogorov complexity K(x)

An important modification of U: if σ, τ are in the domain of U, then τ does not extend σ.

The halting probability of U is

$$\Omega = \sum \{2^{-|\sigma|} : U(\sigma) \text{ halts}\}$$

K(x) is the length of a shortest σ such that U(σ) = x.



If U has access to an oracle set A we write $K^A(x)$

Far-from-random sequences

A is K-trivial if for some number b,

$K(A|n) \leq K(n) + b$ for each n (written in binary).

FACT: If A is computable, then A is K-trivial.

Solovay 1975:

Some A is K-trivial but not computable.

A is K-trivial ↔ A is low for ML-randomness

N., 2002 — every ML-random set is ML-random in A

↔ A is low for K

N., 2002

$$\forall z\, K^A(z) =^+ K(z)$$

↔ A is Turing below some Z that is ML-random in A

Hirschfeldt et al., 2006

↔

… (12 more)

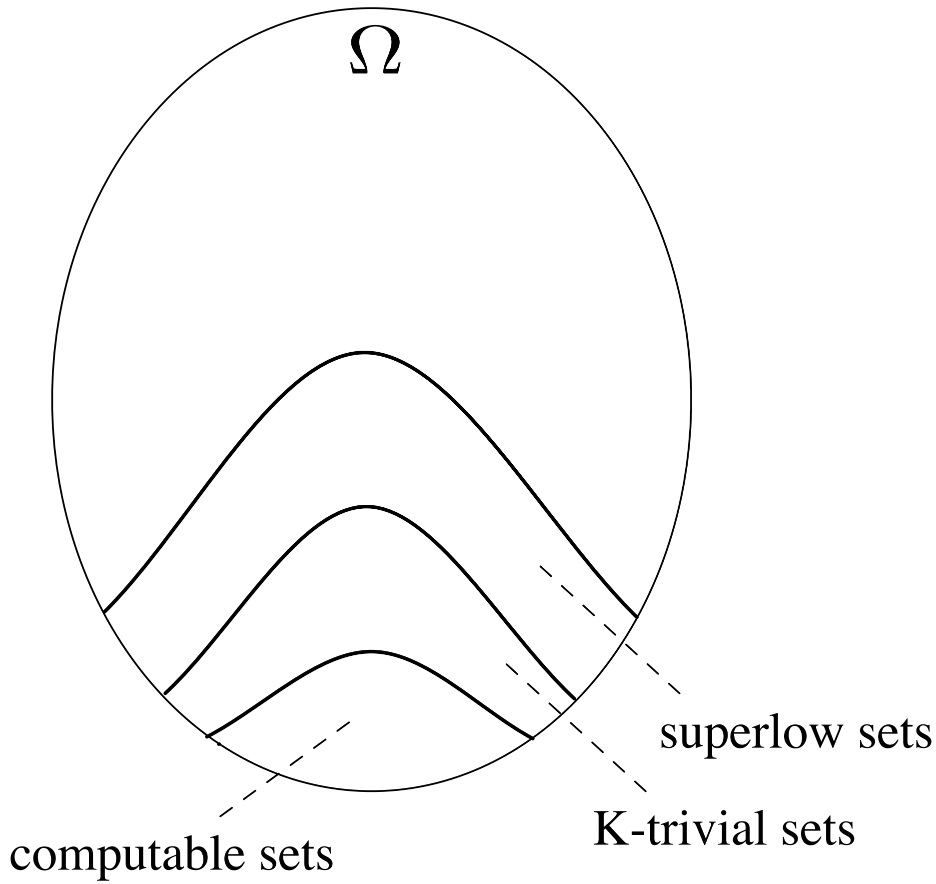
↔ For each Y, if Ω is Y-random then Ω is Y+A-random

Greenberg et al., 2015

Superlowness

An oracle A is called superlow if A' (the halting problem relative to A) is truth table equivalent to Ø'. This is stronger than lowness $A' \equiv_T \emptyset'$.

N. 2005: Every K-trivial set is superlow.



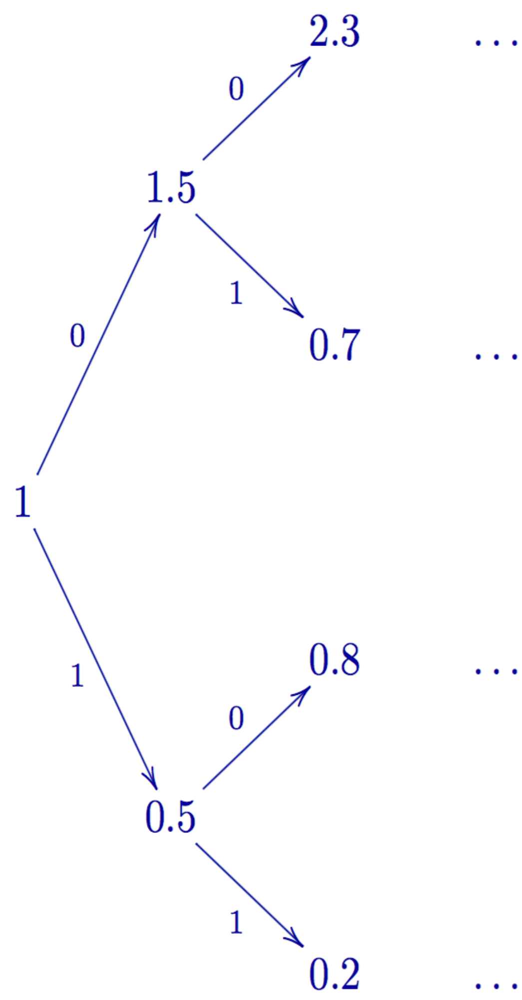
Formalising randomness

A computable betting strategy M makes a guess for the next bit, betting an amount b of the available capital.

The bit is then revealed. If the guess was right, M gets b, otherwise M loses b.

Success of M on a bit sequence Z means the capital along Z is unbounded.

Z is betting random (Schnorr, 1975) if no computable betting strategy succeeds on Z.



Examples of computable betting strategies

- Some betting strategy can win on a sequence with asymptotically at least ε more 1s than 0s

- Another betting strategy can win on a sequence with infinitely many known 0s, e.g., 0 in each position of the form n!

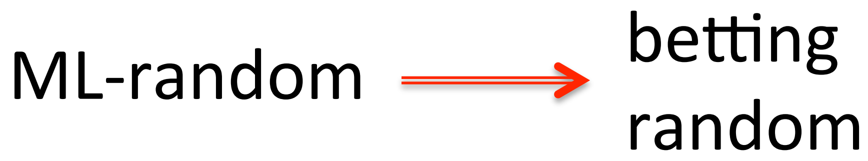
Left-c.e. reals and betting strategies

A real β is called left-computably enumerable (left-c.e.) if $\beta = \sup_s \beta_s$ for some computable sequence of rationals $\langle \beta_s \rangle_{s \in \mathbb{N}}$.

A betting strategy M is called left-c.e. if $M = \sup_s M_s$ for some uniformly computable sequence of $\langle M_s \rangle_{s \in \mathbb{N}}$ of betting strategies.

A bit sequence Z is Martin-Löf random if no left-c.e. betting strategy succeeds along Z.

Diagram of randomness notions so far



Analysis ⟵ Randomness

Effective forms of “almost everywhere” theorems in analysis correspond to randomness notions.

Lebesgue's theorem

Henri Lebesgue introduced a notion of measure (size) for certain sets of real numbers.

Measure on [0,1] can be used to express that a statement holds with probability one.

Lebesgue, 1904: Let f be an increasing function with domain [0,1].
Then f'(z) exists at a real z with probability 1.

Algorithmic forms of Lebesgue's theorem I

A real number z is called betting-random if no effective betting strategy succeeds on the binary expansion of the real.

A function f defined on [0,1] is computable if from a rational approximation to x we can compute one to f(x).

Brattka, Miller, N., 2011 (Trans. AMS, 2016):

A real z is betting random if and only if

f'(z) exists for each increasing

computable function f.

Algorithmic forms of Lebesgue's theorem II

A function $f\colon [a,b] \to \mathbb{R}$ is of bounded variation if

$$V(f) = \sup \sum_{i=1}^{n-1} |f(t_{i+1}) - f(t_i)| < \infty,$$

the sup taken over all collections $t_1 \leq t_2 \leq \ldots \leq t_n$ in $[a,b]$.

Demuth 1975/ Brattka, Miller, N., 2011

A real z is ML-random if and only if

f'(z) exists for each computable function f

of bounded variation.

Algorithmic forms of Lebesgue's theorem III

A real number z is called polynomial time betting random if no polynomial time betting strategy succeeds on the binary expansion of the real.

N., STACS 2014:

A real z is p.t.b. random if and only if

f'(z) exists for each increasing

polynomial time function f

Lebesgue Density Theorem

Let λ denote Lebesgue's measure on [0,1].

Let *E* be a measurable subset of [0,1].

The lower density of *E* at *z* is

$$\underline{\rho}(E \mid z) = \liminf_{z \in J\,,\,|J| \to 0} \frac{\lambda(J \cap E)}{|J|},$$

where J ranges over intervals. This measures how much of E is near z, as one zooms in on z.

Lebesgue, 1910:

Let *E* be measurable. For almost every *z* in *E*, the set *E* has lower density *1* at *z*.

A ML-random failing effective Lebesgue Density Theorem

Recall: Let E be measurable. For almost every z in E, the set E has lower density *1* at z.

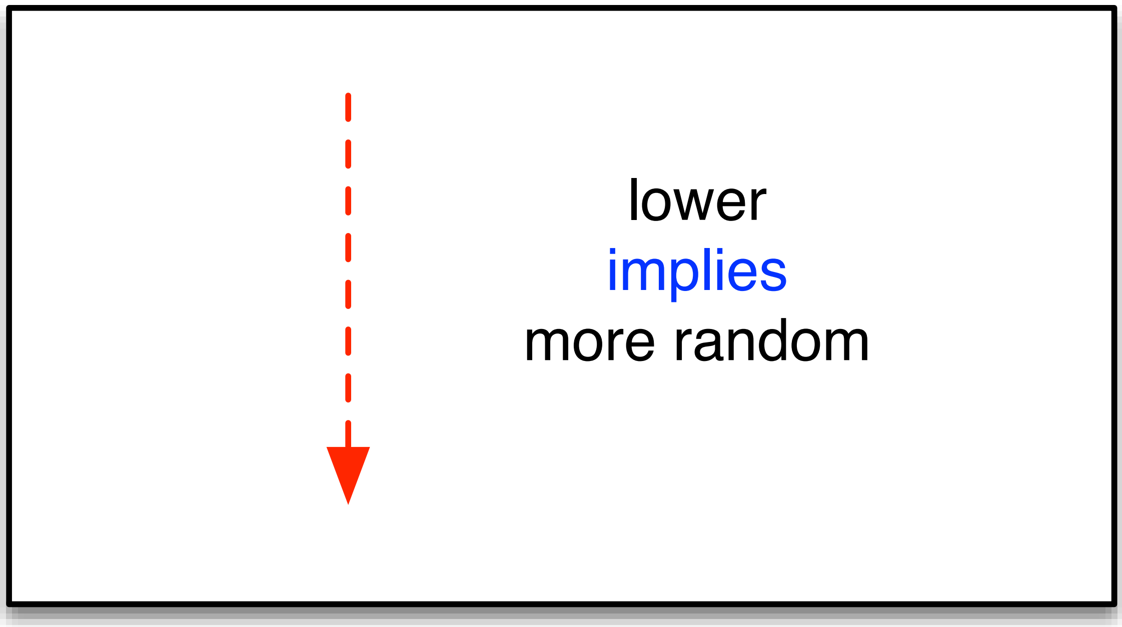
- Ω denotes Chaitin's halting probability, a ML-random real that is left-computably enumerable.
- Let E= [Ω,1]. Then E is an effectively closed set.
- Ω is in E but the lower density of E at Ω is 0.

Lowness ←—— Randomness

Lowness calibrates randomness notions.

Randomness increases with lowness

given randomness notion



- If Z is betting random and not high, then Z is ML-random.
- If Z is ML-random and forms a Turing minimal pair with the halting problem, then Z is weakly 2-random.

Randomness increases with lowness

Franklin and Ng (2012) introduced difference randomness via an algorithmic test notion. They showed that it coincides with: ML random and not Turing above the halting problem.

Difference tests were later emulated by density:

Bienvenu, Hölzl, Miller, N., JML 2014:

Let z be ML-random. Then z is not above the halting problem (computationally weak) iff every effectively closed E containing z has positive lower density at z (a stronger randomness condition on z).

A difference random failing effective Lebesgue Density Theorem

Day and Miller, 2014

There is a ML-random Z below the halting problem such that

- every effectively closed set containing Z has positive lower density at Z
- some effectively closed set containing Z has lower density less than 1 at Z.

Thm. of Madison group + Myabe-N.-Zhang

The following are equivalent for a real z with binary expansion 0.Z:

- z is ML-random and every effectively closed set E containing z has density 1 at z (z is density random)
- every left-c.e. betting strategy converges along Z
- f'(z) exists for every function f that is the variation of a computable function
- for each lower semicomputable integrable g we have

$$g(z) = \lim_{Q \to z} \frac{1}{|Q|} \int_Q g \, d\lambda$$

where Q ranges over intervals containing z.

The Madison group (2012) consisted of Andrews, Diamondstone, Lempp, Miller, and Ng. This theorem is published in M-N-Z 2016 BSL paper.

Diagram of randomness notions (growing)

density random → difference random → ML-random → betting random

every effectively closed set E containing z has density 1 at z

every effectively closed set E containing z has positive density at z

Tests given by effectively open sets

A subset N of [0,1] is a null set iff for each ε, there is an open set G containing N with λ(G) ≤ ε.

A test is a descending sequence $\langle G_m \rangle_{m\in\mathbb{N}}$ of sets that are effectively open uniformly in *m*, such that $\lim_m \lambda(G_m) = 0.$

Effectively open means:
union of an effective list of rational open intervals.

Left-c.e. tests and Martin-Löf test

Effectivise condition that $\lim_m \lambda(G_m) = 0.$

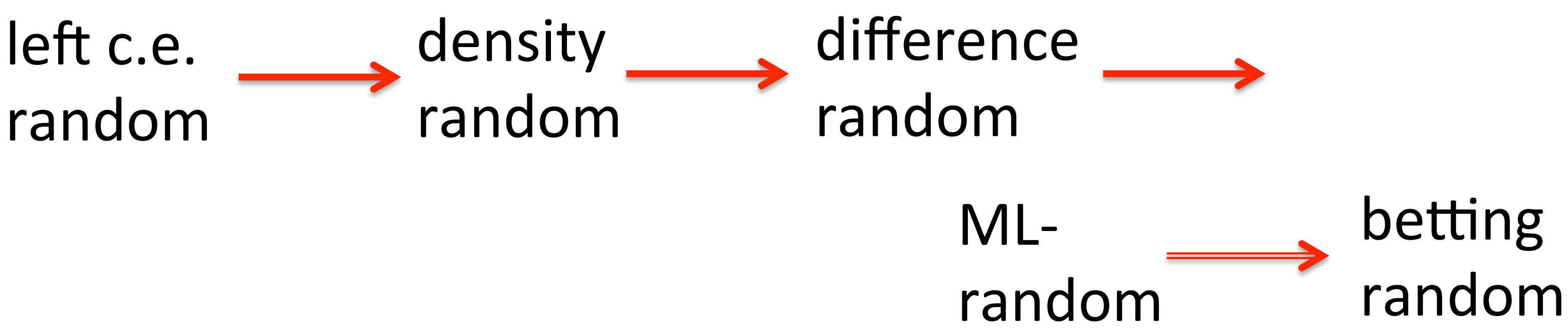
Left-c.e. test: $\lambda(G_m) \leq \beta - \beta_m$

where $\langle \beta_m \rangle$ is computable sequence of rationals with $\beta = \sup_m \beta_m < 1.$

Martin-Löf test: $\lambda(G_m) \leq 2^{-m}.$

z is ...- random if $z \notin \bigcap_m G_m$ for any test of this kind.

Diagram of randomness notions (finished)



Is left-c.e. random = density random?

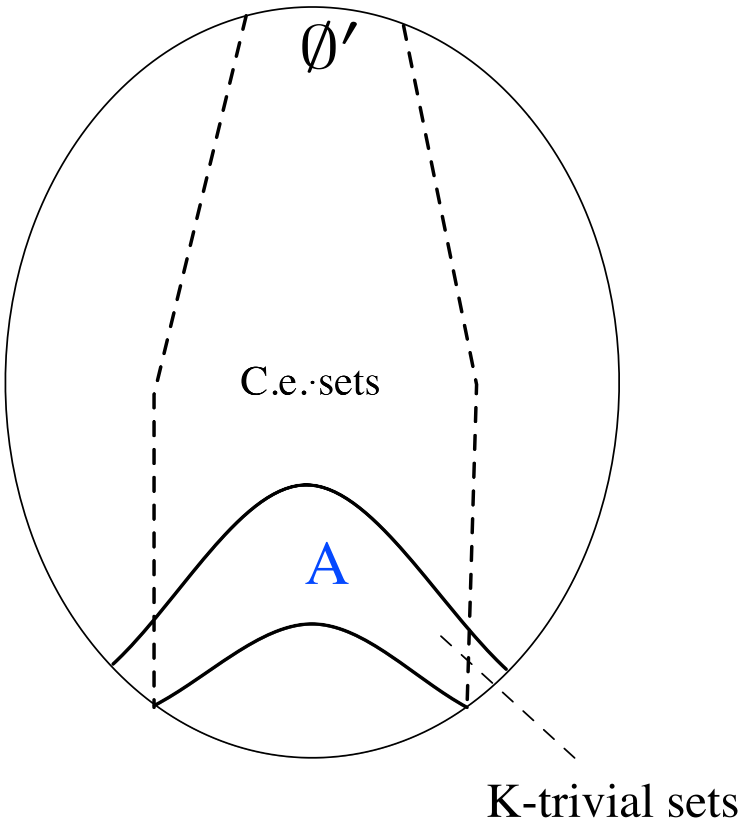
Analysis ← Randomness

Randomness ← Lowness

New insights of analysis about randomness help to study lowness.

Covering problem

Stephan 2004; Miller-N.: Turing above each K-trivial A, is there an incomplete ML-random Z?



Covering problem

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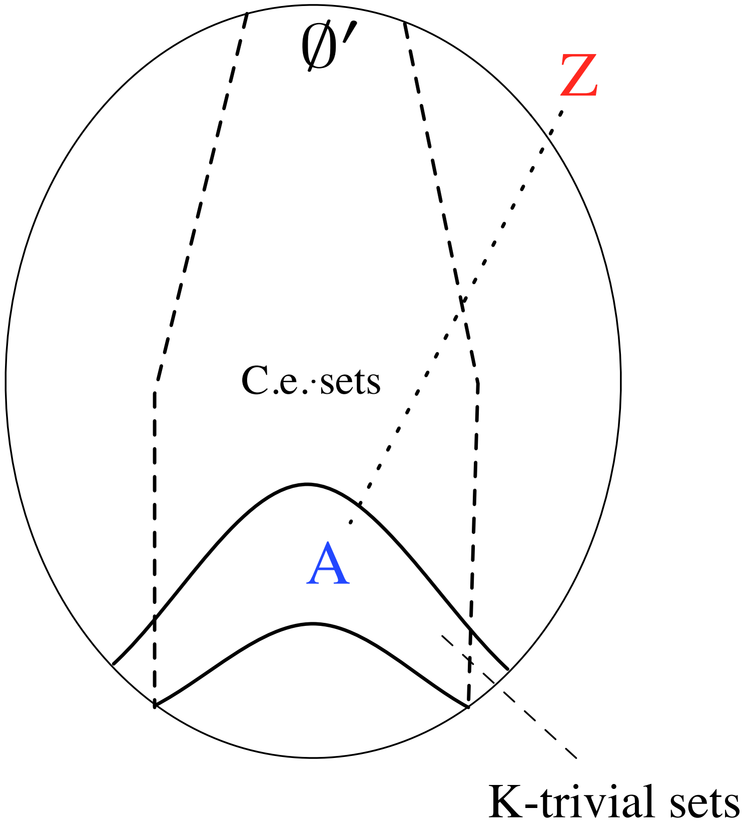


This asked for a converse to known result (Hirschfeldt et al. 04):

If A is c.e. and below a Turing incomplete ML-random Z, then A is K-trivial.

Covering problem

Stephan 2004; Miller-N. 2006: Turing above each K-trivial A, is there an incomplete ML-random Z?

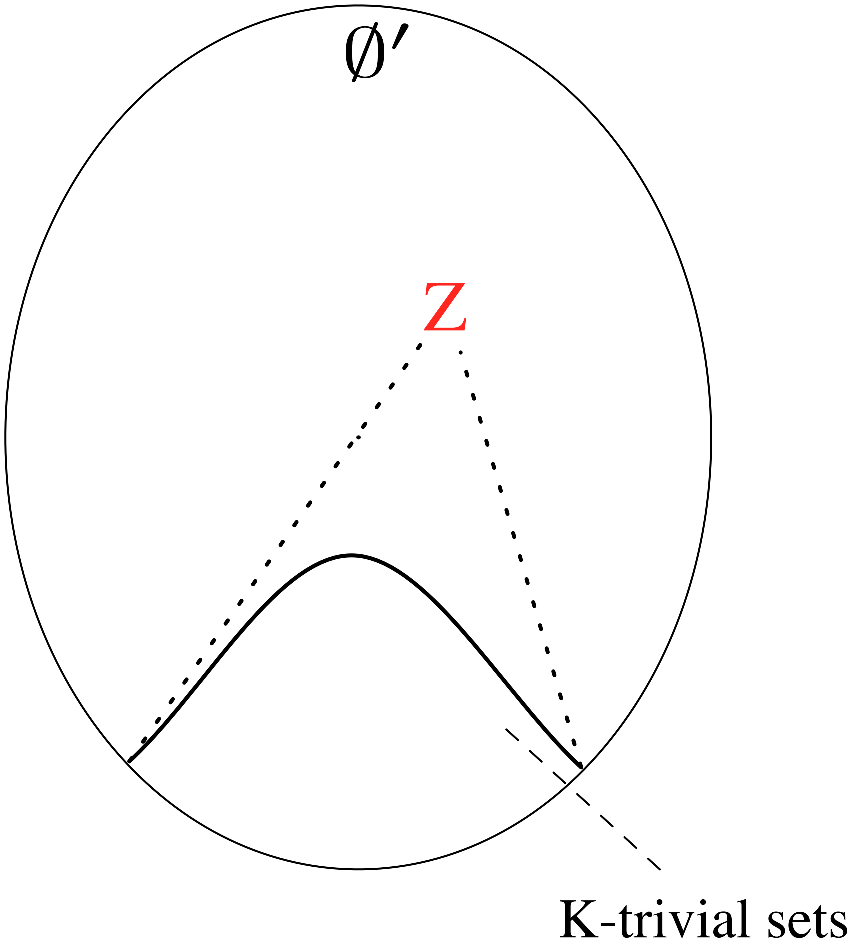
Day- Miller theorem on density, and the following two results were needed to resolve the covering problem.

Bienvenu, Greenberg, Kucera, N., Turetsky:

(1) Let Z be ML-random and not left-c.e. random. Then Z is Turing above all the K-trivials.

(2) If Z is left-c.e. random, then Z is density random.

Strong solution to the covering problem



Day and Miller showed that some Turing incomplete ML-random Z is not density random.

Combining this with the results of the OW group, Z is not left-c.e. random, so it is Turing above all the K-trivials.

Is left-c.e. random = density random?

Analysis

Lowness

Concepts from analysis
parameterise lowness notions.

Parameterising lowness notions I:

cost functions

Definition of cost function

A real β≥0 is called left-computably enumerable (left-c.e.) if $\beta = \sup_s \beta_s$ for some computable sequence of rationals $\langle \beta_s \rangle_{s\in\mathbb{N}}$.

A cost function is a sequence $\langle c(x) \rangle_{x\in\mathbb{N}}$ of uniformly left-c.e. reals that converges to 0.

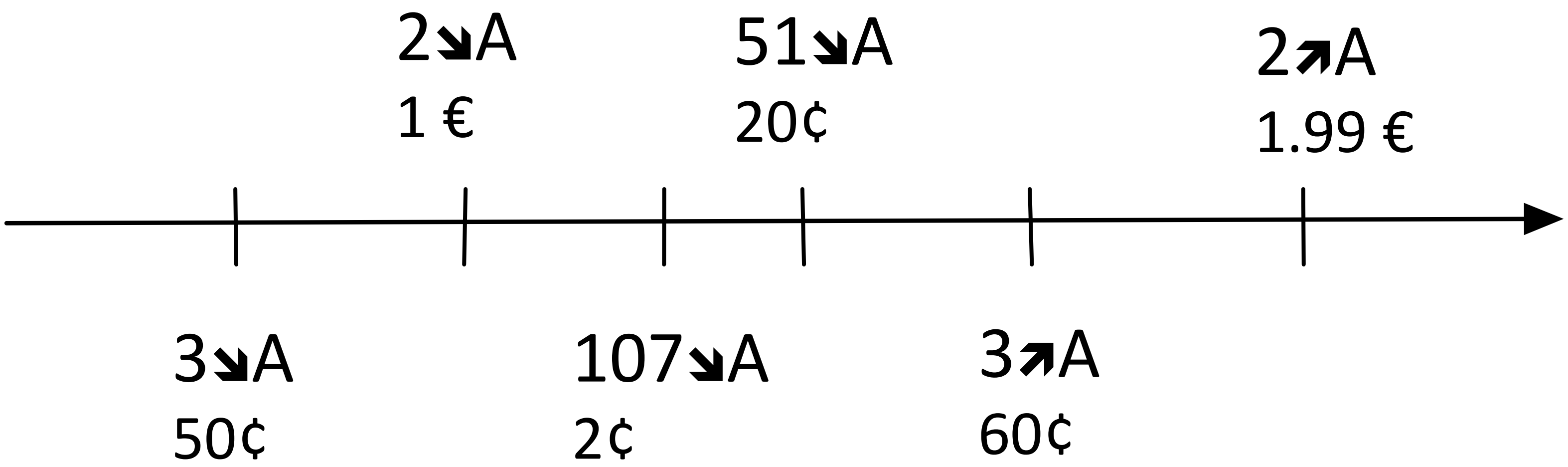
Example: $$c_\Omega(x) = \Omega - \Omega_x.$$

Here Ω_x is the measure of U-computations that have converged by stage x.

Obedience to cost functions

When building a computable approximation $A(x) = \lim_s A_s(x)$, we view $c_s(x)$ as the cost of changing A(x) at stage s.

A obeys c if for some approximation, the changes over all stages s have a finite total cost. (We only count the least change at each stage.)



Existence theorem

Recall: a set A obeys a cost function c if for some computable approximation, the changes over all stages s have a finite total cost.

Thm: For each cost function c there is a c.e., noncomputable set A that obeys c.

Proof idea: If at stage s we see an $x \geq 2e$ in the e-th c.e. set W_e such that $c(x, s) \leq 2^{-e}$ put x into A.
Some "bookkeeping" is needed.

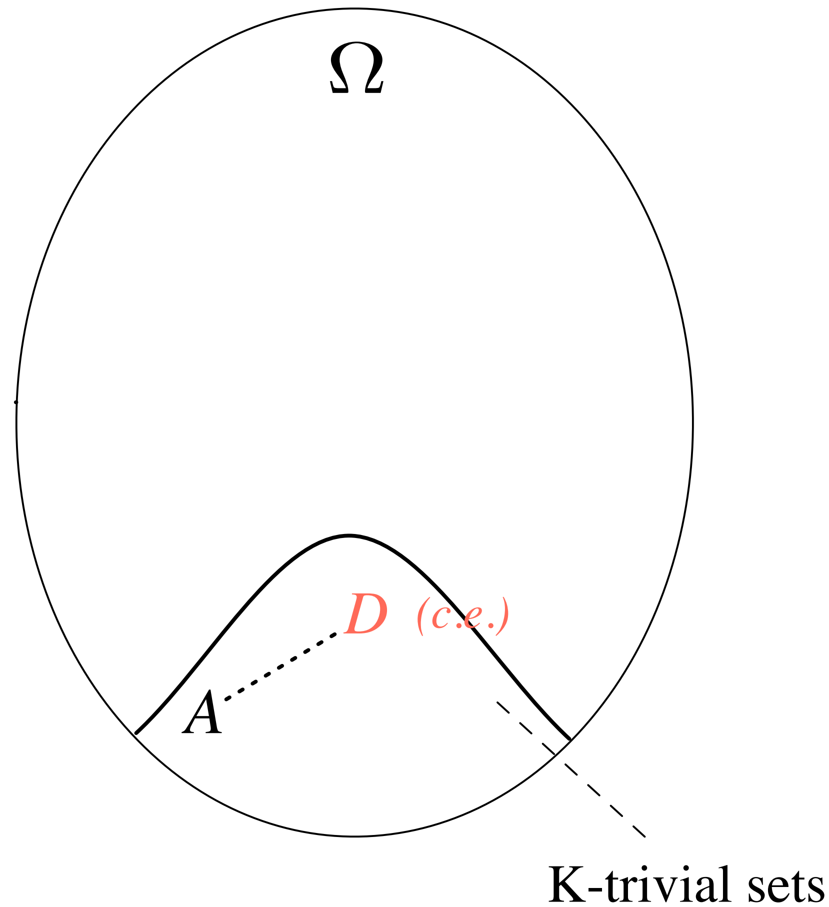
Cost function characterisation of K-triviality

Recall: $$c_\Omega(x) = \Omega - \Omega_x.$$

Thm (N. 2011) : A is K-trivial iff A obeys c_Ω.

Earlier version 2005 with a different (less neat) cost function.

As a consequence, every K-trivial A is truth-table below some c.e. K-trivial (N., 2005)



A dense hierarchy of ideals of K-trivials

Cost functions parameterise a dense hierarchy of subideals of the K-trivials. For a positive rational q< 1 let $c_{\Omega,q}(x) = (\Omega - \Omega_x)^q$.

Thm. (Greenberg, Miller and N. 2015)

Let q =k/n<1. Then a set A is Turing below every k out of n "columns" of some ML-random oracle Z iff A obeys $c_{\Omega,q}$.

Ideal given by $c_{\Omega,q}$ gets bigger as q increases.

Tests bounded by cost functions

Let c be a cost function.
A test $\langle G_m \rangle_{m\in\mathbb{N}}$ is called a c-test if

$$\lambda(G_m) = O(c(m))$$

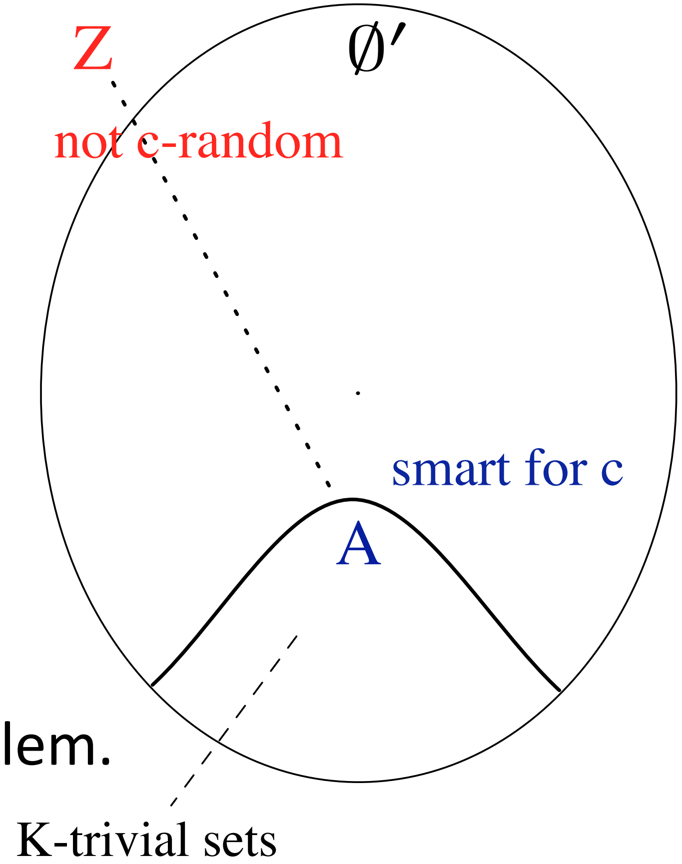
Z is c-random if Z passes each c-test.

Left-c.e. random is the same as c_Ω-random.
If A obeys c then A is Turing below each ML-random that fails to be c-random.

Smart sets for cost functions

Bienvenu et al. (now in JEMS, 2016) built a c.e. K-trivial A such that no set Z above A is left-c.e. random. This was generalised to all larger cost functions in place of c_Ω :

Thm. (Greenberg, Miller, N. and Turetsky)
For each cost fcn $c \geq c_\Omega$ there is a c.e. set A obeying c such that no set Z above A is c-random.



The construction of A provides a requirement-free solution to Post's problem.

Parameterising lowness notions II:

Hausdorff distance to the computable sets

The upper density of a bit sequence Z is

$$\overline{\eta}(Z) = \limsup_n \frac{|Z \cap [0,n)|}{n}.$$

Besicovich pseudo-distance on Cantor space:

$$d(U,W) = \overline{\eta}(U \triangle W).$$

The Hausdorff distance between the Turing cone below A and the computable sets measures how close A is to computable.

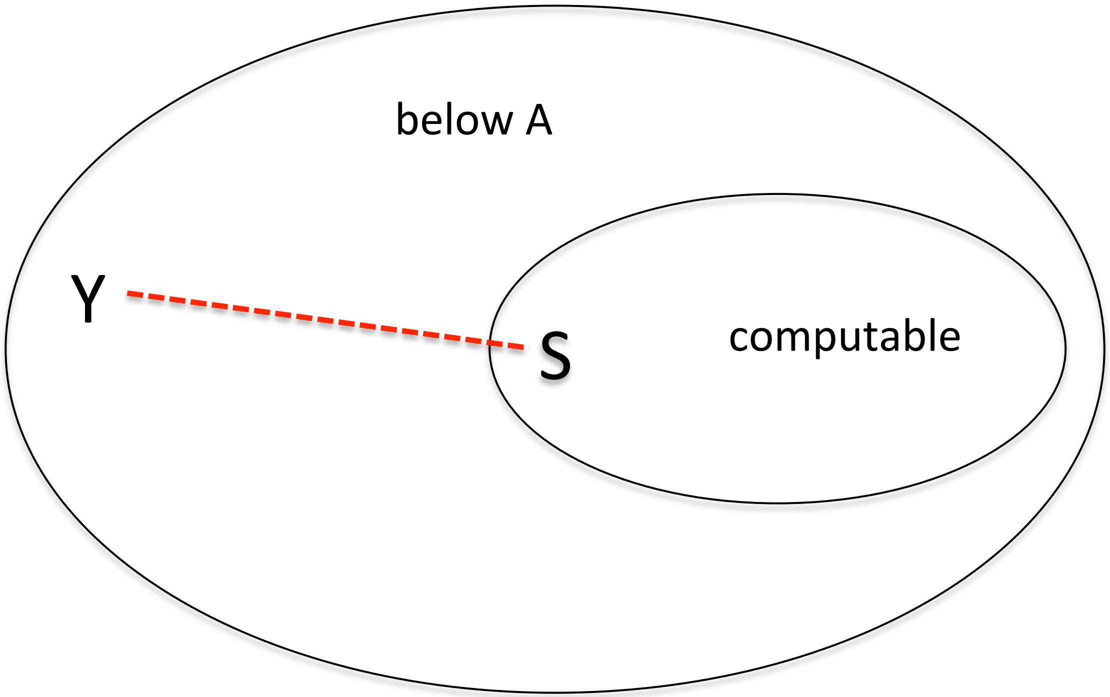
Let $\mathcal{A} = \{Y : Y \leq_T A\}$ and $\mathcal{R}$ the computable sets.

$$d_H(\mathcal{A},\mathcal{R}) = \sup_{Y \in \mathcal{A}} \inf_{R \in \mathcal{S}} d(Y,R)$$

Hausdorff distance gauges lowness of A

$\mathcal{A} = \{Y : Y \leq_T A\}$ $\mathcal{R}$ = computable

$$d_H(\mathcal{A}, \mathcal{R}) = \sup_{Y \in \mathcal{A}} \inf_{S \in \mathcal{R}} d(Y, S)$$ Hausdorff distance.



Values: 0, ½, 1, and no others

$$L(A) = d_H(\mathcal{A}, \mathcal{R}) = \sup_{Y \in \mathcal{A}} \inf_{S \in \mathcal{R}} d(Y, S)$$

Andrews et al., 2013:

- L(A) < ½ iff L(A)=0 iff A is computable
- L(A)= ½ is possible
- L(A)= 1 is possible.

Values: 0, ½, 1, and no others

$$L(A) = d_H(\mathcal{A}, \mathcal{R}) = \sup_{Y \in \mathcal{A}} \inf_{S \in \mathcal{R}} d(Y, S)$$

Monin, Nies, LICS 2015:

If every Schnorr test relative to A is passed by a computable set, then L(A)= ½.

If A computes a function $< 2^{2^n}$ that is infinitely often equal to each computable function, then L(A) = 1

Monin:

L(A)> ½ implies L(A)= 1, and is equivalent to the latter condition. This solved the "Gamma question".

Conclusion

Lowness and randomness interact in both directions: lower means more random, K-trivials

Analysis can be used to study randomness (via differentiability and density), and indirectly lowness (solution to covering problem)

Analytic concepts can also be used to directly calibrate lowness (cost functions, L(A))

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