Lightface Π^0_3-completeness of density sets under effective Wadge reducibility

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Some definitions

- $2^{\mathbb{N}}$ is Cantor space, μ the uniform product measure.
- For a finite bit string s, by $[s]$ we denote the clopen set in $2^{\mathbb{N}}$ consisting of all the extensions of s.
- For a measurable set $\mathcal{A} \subseteq 2^{\mathbb{N}}$, the local measure of $\mathcal{A}$ above s is
$$\mu_s(\mathcal{A}) = \mu([s] \cap \mathcal{A})/2^{-|s|}.$$
- The density set of $\mathcal{A}$ is
$$D\mathcal{A} = \{Z\colon \lim_n \mu_{Z\restriction n}(\mathcal{A}) = 1\}$$
- $D\mathcal{A} = \{Z\colon \forall \epsilon > 0 \exists k \forall n \geq k \, \mu_{Z\restriction n}(\mathcal{A}) \geq 1 - \epsilon\}$ so this set is $\mathbf{\Pi}^0_3$.
- Lebesgue density theorem (1904): $\mathcal{A} \triangle D\mathcal{A}$ is null.

Complexity analysis in descriptive set theory

Let $\mathcal{C}, \mathcal{D} \subseteq 2^{\mathbb{N}}$. One says that $\mathcal{D}$ is Wadge below $\mathcal{C}$ if $\mathcal{D} = F^{-1}(\mathcal{C})$ for a continuous function $F\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

Our main result is an analog in computability theory of the following recent theorem from descriptive set theory.

Theorem (Andretta-Camerlo, Advances in Math 2013)

Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be measurable set with empty interior such that $\mu\mathcal{A} > 0$. The density set of $\mathcal{A}$ is (boldface) $\mathbf{\Pi}^0_3$-complete for Wadge reductions.

The hypotheses are necessary:
if e.g. $\mathcal{A} = 2^{\mathbb{N}}$ then $D\mathcal{A} = \mathcal{A}$; if $\mu\mathcal{A} = 0$ then $D\mathcal{A} = \emptyset$.

Completeness for lightface point classes

Effective version of Wadge reducibility: for $\mathcal{C}, \mathcal{D} \subseteq 2^{\mathbb{N}}$, $\mathcal{D} \leq_m \mathcal{C} \Leftrightarrow \mathcal{D} = \Psi^{-1}(\mathcal{C})$ for some total Turing functional (i.e., truth table functional) Ψ.

Definition

Let Γ be a lightface point class in Cantor space, such as Π^0_2 or Π^0_3. We say that $\mathcal{C} \in \Gamma$ is Γ-complete (for effective Wadge reductions) if for each $\mathcal{D} \in \Gamma$ we have $\mathcal{D} \leq_m \mathcal{C}$.

Examples of completeness

Proposition (Folklore)

The class $\mathcal{C}$ of all sequences with infinitely many 1's is Π^0_2-complete.

- We effectively identify $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ with $2^{\mathbb{N}}$ via a computable pairing function.
- We will need the following for our main result. It says that "every column is eventually 0" is Π^0_3-complete.

Proposition

The set $\mathcal{E} = \{Z \in 2^{\mathbb{N}} \colon \forall q \forall^\infty n \, Z(q,n) = 0\}$ is Π^0_3-complete.

Π^0_3 completeness of the density set

$D\mathcal{A} = \{Z\colon \lim_n \mu_{Z\upharpoonright n}(\mathcal{A}) = 1\}$. If $\mathcal{A}$ is closed then $D\mathcal{A} \subseteq \mathcal{A}$.

Theorem (Main)

Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be a Π^0_1 set with empty interior such that $\mu\mathcal{A} > 0$ and $\mu\mathcal{A}$ is a computable real. Then $D\mathcal{A}$, the density set of $\mathcal{A}$, is Π^0_3-complete for effective Wadge reductions.

Example of such $\mathcal{A}$: let $n_1 = 0$. For $k > 0$ let $n_{k+1} = n_k + k$.

$$\mathcal{A} = \{Z\colon \forall k\,[Z \upharpoonright [n_k, n_{k+1}) \neq \overbrace{0\ldots0}^{k}]\}.$$

$0 < \mu\mathcal{A} = \prod_{k>0}(1 - 2^{-k})$ is computable. For $Z \in \mathcal{A}$ let

$$f_Z(k) = k - \text{ first position of a 1 in } [n_k, n_{k+1}).$$

$$\boxed{D\mathcal{A} = \{Z \in \mathcal{A}\colon\ \lim_k f_Z(k) = \infty\}.}$$

By the theorem, this density set is Π^0_3 complete.

Check that $D\mathcal{A}$ is Π^0_3

$\mu_s(\mathcal{A}) = \mu([s] \cap \mathcal{A})/2^{-|s|}$.
$D\mathcal{A} = \{Z \colon \lim_n \mu_{Z\restriction n}(\mathcal{A}) = 1\}$.

Theorem (recall)

Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be a Π^0_1 set with empty interior such that $\mu\mathcal{A} > 0$ and $\mu\mathcal{A}$ is a computable real. Then $D\mathcal{A}$, the density set of $\mathcal{A}$, is Π^0_3-complete for effective Wadge reductions.

- $Z \in D\mathcal{A} \leftrightarrow \forall \epsilon \in \mathbb{Q}^+ \exists n \forall k \geq n\, [\mu_{Z\restriction k}(\mathcal{A}) \geq 1 - \epsilon]$.
- Since $\mathcal{A}$ is Π^0_1, the statement $\mu_z(\mathcal{A}) \geq 1 - \epsilon$ is Π^0_1 uniformly in ϵ and a string z.

Proof sketch for the Π^0_3 completeness

$$L(s) = \mu_s(\mathcal{A}) \text{ (that is, the local measure of } \mathcal{A} \text{ in } [s]).$$

L is a computable betting strategy (martingale) in the sense of algorithmic randomness:

- $\forall s\, L(s0) + L(s1) = 2L(s)$
- $L(s) \geq 0$ is a computable real uniformly in s, since μA is computable

Also $L(s) < 1$ for each s because $[s] \not\subseteq \mathcal{A}$. We have $Z \in D\mathcal{A} \leftrightarrow \lim_n L(Z \upharpoonright n) = 1$.

Idea:
Oscillation of L along paths can be controlled sufficiently well in order to code the Π^0_3-complete set $\mathcal{E}$ into $D\mathcal{A}$.

- Recall $L(s) = \mu_s(\mathcal{A})$. For $p \in \mathbb{N}$ let $\delta_p = 1 - 3^{-p}$.
- Let $\theta(s) = \begin{cases} p & \text{if } \delta_{p-1} < L(s) < \delta_p \\ \uparrow & \text{if } \exists k \, L(s) = \delta_k \end{cases}$

θ is partial computable. $\lim_n \theta(X \upharpoonright n) = \infty \Rightarrow X \in D\mathcal{A}$.

Lemma (modifying Andretta/Camerlo)

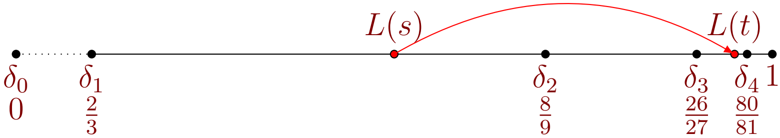
(i) (Increasing the value of L.) *Let* $p < k$ *and* $\theta(s) = p$. *There is* $t \supset s$ *such that* $\theta(t) \geq k$ *and* $L(u) > \delta_{p-1}$ *for each* u *with* $s \subseteq u \subseteq t$.

(ii) (Decreasing the value of L.) *Let* $p > q$ *and* $\theta(s) = p$. *There is* $t \supset s$ *such that* $\theta(t) = q$ *and* $L(u) > \delta_{q-1}$ *for each* u *with* $s \subseteq u \subseteq t$.

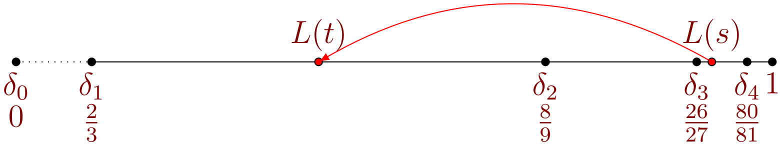
$L(x)$ is a computable real uniformly in string x. Hence all conditions in the Lemma are Σ^0_1. So we can search for t.

Examples

(i) Increasing the value of L.
Suppose $\theta(s) = p = 2$ and $k = 4$.



(ii) Decreasing the value of L.
Suppose $\theta(s) = p = 4$ and $q = 2$.



Defining the tt-reduction ψ by recursion.

We want to show

$$\mathcal{E} = \{Z \in 2^{\mathbb{N}} \mid \forall q \forall^\infty n [Z(q,n) = 0\} \leq_m D\mathcal{A} \text{ via } \psi.$$

We may assume $\mu\mathcal{A} \leq 1/2$, so $\theta(\emptyset)$ is defined. Let $\psi(\emptyset) = \emptyset$.
Given a matrix $b = \langle b_{i,j} \mid i,j < n+1 \rangle$, let $s = \psi(a)$ where $a = b \restriction n \times n$. $p = \theta(s)$ is defined.
If there is a $q \leq n$ such that $b(q,n) = 1$, choose q least.

- $\boxed{\text{If } q < p\text{, go down:}}$ Compute a string $t \supset s$ such that $\theta(t) = q$ and $L(u) > \delta_{q-1}$ for each u with $s \subseteq u \subseteq t$, and define $\psi(b) = t$.
- $\boxed{\text{Otherwise go up:}}$ let $k = \max(p+1, n+1)$. Compute a string $t \supset s$ such that $\theta(t) \geq k$ and $L(u) > \delta_{p-1}$ for each u with $s \subseteq u \subseteq t$, and define $\psi(b) = t$.

This completes the recursion step.

Remarks and open questions

Fact

The hypothesis that $\mu\mathcal{A}$ be computable is needed to obtain the Π^0_3-completeness of $D\mathcal{A}$.

For instance, take a nonempty Π^0_1 class $\mathcal{A}$ of Martin-Löf randoms; $\mathcal{A}$ has empty interior, but its measure $\mu\mathcal{A}$ is a ML-random real. ($D\mathcal{A}$ is complex in a sense- e.g each density random is in $D\mathcal{A}$, but the right-c.e. real $\min\mathcal{A}$ is not.) Note that $\mathcal{E}$ contains computable sets, e.g. the empty set. Since $\mathcal{A}$ and hence $D\mathcal{A}$ do not contain computable sets, we have $\mathcal{E} \not\leq_m D\mathcal{A}$.

Question

What is the complexity of $D\mathcal{B}$ for a Π^0_2-class $\mathcal{B}$?

Note that $D\mathcal{B}$ is Π^0_3 relative to $\emptyset'$.