The Gamma question and cardinal characteristics

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The Gamma question and its variants

Lowness paradigms

Given a set $A \subseteq \omega$. How close is A to being computable?

Several paradigms have been suggested.

- A has little power as a Turing oracle.
  E.g. computably dominated: every function $g \leq_{\mathrm{T}} A$ is dominated by a computable function.
- Many oracles compute A.
  E.g. base for randomness: there is an oracle Z, random in A, such that $Z \geq_T A$.

A recent paradigm: A is coarsely computable. This means there is a computable set R such that the asymptotic density of $A \leftrightarrow R$ equals 1. Here $A \leftrightarrow R := \{n \colon A(n) = R(n)\}$.

Downey, Jockusch, and Schupp, Asymptotic density and computably enumerable sets, Journal of Mathematical Logic, 13, No. 2 (2013)

The γ-value of a set $A \subseteq \omega$

The lower density of $Z \subseteq \omega$ is a number in $[0, 1]$, namely

$$\underline{\rho}(Z) = \liminf_n \frac{|Z \cap [0, n]|}{n}$$

For the γ value, smaller means harder to compute:

$$\gamma(A) = \sup_{R \text{ computable}} \underline{\rho}(A \leftrightarrow R).$$

- If $\gamma(A) \geq p$ one says that
  A is coarsely computable at density p.
- A coarsely computable $\Rightarrow \gamma(A) = 1$.
  Hirschfeldt et al. show that the converse fails.

Hirschfeldt, Jockusch, McNicholl, Schupp, *Asymptotic density and the coarse computability bound*, preprint, 2013.

Possible γ values

A computable: 1. A random: $1/2$

Any computable (in fact, left- Σ^0_2) real in $[0,1]$ is a $\gamma(A)$.

Fact

Let A be not computably dominated, i.e.

$\exists\, g \leq_{\mathrm{T}} A$ not dominated by any computable function h.

There is $B \leq_{\mathrm{T}} A$ such that $\gamma(B) = 0$.

Proof idea: for $n > 0$ let $I_n = [(n-1)!, n!)$.

- For a computable set R given by characteristic function Φ_e, let $h(n)$ be the time it takes Φ_e to converge on I_n.
- If $h(n) < g(n)$, A sees this convergence in time, and can make B different from R on all of I_n. This ensures

$$\frac{|(B \leftrightarrow R) \cap [0, n!)|}{n!} \leq 1/n.$$

- Do some bookkeeping to treat all the total Φ_e.

  For such e there are infinitely many n with $h(n) < g(n)$.

Γ-value of a Turing degree

Andrews et al. (2013) wanted to look at degrees, rather than sets. So they defined

$$\Gamma(A) = \inf\{\gamma(B)\colon\ B \equiv_T A\}.$$

One can as well take the inf over all $B \leq_{\mathrm{T}} A$.

- By previous fact, if A is not computably dominated then $\Gamma(A) = 0$.
- They show: if A is random and computably dominated, then $\Gamma(A) = 1/2$.

Andrews, Cai, Diamondstone, Jockusch and Lempp, *Asymptotic Density, computable traceability, and 1-randomness*, preprint, 2013.

How about Γ values in $(1/2, 1)$?

Fact (Hirschfeldt et al.)

If $\Gamma(A) > 1/2$ then A is computable (so that $\Gamma(A) = 1$).

Proof idea: Again let $I_n = [(n-1)!, n!)$ for $n > 0$.

- Define $B(k) = A(n)$ for $k \in I_n$. Then $B \equiv_T A$.
- Suppose that $\gamma(R \leftrightarrow B) > 1/2$ for a computable R.
- Then for almost all n
  (namely, all n with $1/n < \Gamma(A) - 1/2$),
  $A(n)$ is the value of the majority of bits $R(k)$ for $k \in I_n$.
- This we can compute.

Γ-question, Andrews et al., 2013

We have the Γ values $1/2$, 1 and nothing properly in between.
We also have the value 0.
Is there any value properly in between 0 and 1?

Question (Γ-question)

Is there an A such that $0 < \Gamma(A) < 1/2$?

- We won't answer this.
- Rather, we use analogs of notions from cardinal characteristics to obtain natural classes of oracles with Γ value $1/2$, and with Γ value 0.
- This yields new examples for both cases.
- It may help to solve the problem eventually, by providing methods and a conceptual framework.

Variants of γ and Γ

Other bases: We can work in base $b > 2$ rather than 2, i.e. $A\colon \omega \to \{0, \dots, b-1\}$.
$\gamma_b(A)$ and $\Gamma_b(A)$ are defined as expected.
We have $\Gamma_b(A) > 1/b \Rightarrow A$ computable (now with a harder proof). Values in $(0, 1/b)$?

Complexity theory: Fix an alphabet Σ. For $Z, A \subseteq \Sigma^*$ let

$$\begin{aligned}
\underline{\rho}(Z) &= \liminf_n \frac{|Z \cap \Sigma^{\leq n}|}{|\Sigma^{\leq n}|} \\
\gamma_{\texttt{poly}}(A) &= \sup_{R \text{ poly time computable}} \underline{\rho}(A \leftrightarrow R) \\
\Gamma_{\texttt{poly}}(A) &= \inf\{\gamma_{\texttt{poly}}(B)\colon\ B \equiv_T^p A\}.
\end{aligned}$$

Not much known here. Basic facts from computability don't carry over. Which $\Gamma_{\texttt{poly}}$ values exist?

Cardinal characteristics

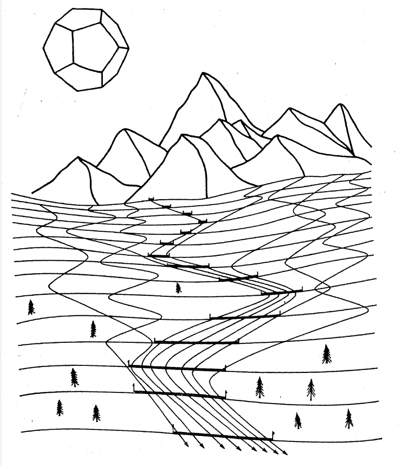
Cardinal characteristics and highness properties

Recent interaction of set theory and computability:
A close analogy between cardinal characteristics of the continuum, and highness properties (indicating strength of a Turing oracle).

- The domination number $\mathfrak{d}$ is the least size of a set of functions on the natural numbers so that every function is dominated by one of them.
- This corresponds to being not computably dominated: the oracle A computes a function that is not dominated by any computable function.
- This correspondence was first studied explicitly by Nicholas Rupprecht, a student of A. Blass (thesis, article in Arch. of Math. Logic, 2010).

Domination, and slaloms (Bartoszyński, 1987)

- For $f, g \in {}^{\omega}\omega$, let $f \leq^* g \Leftrightarrow f(n) \leq g(n)$ for almost all n.
- A slalom is a function σ from ω the finite subsets of ω such that
$$\forall n \, |\sigma(n)| \leq n^2.$$
- It traces f if $\forall^\infty n \, f(n) \in \sigma(n)$.



Picture in Bartoszyński's paper

Set theory versus computability

$\mathfrak{d} = \mathfrak{d}(\leq^*)$: the least size of a set $F \subseteq {}^\omega\omega$ dominating each function.

cofin$(\mathcal{N})$: the least size of a collection of null sets covering all null sets.

$\mathfrak{d}(\in^*)$: the least size of a set of slaloms tracing all functions.

Thm. [Bartoszyński 1984] $\mathfrak{d}(\in^*) = \mathrm{cofin}(\mathcal{N})$.

there is $g \leq_T A$ such that $g \not\leq^* f$ for each computable f.

A is not low for Schnorr tests.

A is not computably traceable.

Thm. [Terw./Zamb. 2001] Computably traceable $\Leftrightarrow$ low for Schnorr tests

Unbounding and domination numbers of relations

Let $R \subseteq X \times Y$ be a relation. Let

$$\begin{aligned}\mathfrak{b}(R) &= \min\{|F| : F \subseteq X \wedge \forall y \in Y \exists x \in F \;\; \neg xRy\} \\ \mathfrak{d}(R) &= \min\{|G| : G \subseteq Y \wedge \forall x \in X \, \exists y \in G \;\; xRy\}.\end{aligned}$$

- $\mathfrak{b}(R)$ is called the unbounding number of R, and $\mathfrak{d}(R)$ the domination number.
- If R is a preordering without greatest element, then any set of covers is unbounded. So ZFC proves $\mathfrak{b}(R) \leq \mathfrak{d}(R)$.

Extended Cichoń diagram of cardinals (10 nodes)

$$\begin{tikzcd}
 & \mathfrak{b}(\neq^*) \arrow[d, equal, "\text{BJ, Th. 2.4.7}"] & & \mathfrak{d}(\in^*) \arrow[d, equal, "\text{BJ, Th. 2.3.9}"] \\
\mathrm{cover}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cofin}(\mathcal{M}) \arrow[r] & \mathrm{cofin}(\mathcal{N}) \\
 & \mathfrak{b}(\leq^*) \arrow[u] \arrow[r] & \mathfrak{d}(\leq^*) \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cover}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] \\
\mathfrak{b}(\in^*) \arrow[u, equal, "\text{BJ, Th. 2.3.9}"] & & \mathfrak{d}(\neq^*) \arrow[u, equal, "\text{BJ, Th. 2.4.1}"] &
\end{tikzcd}$$

$\mathcal{M}$ denotes “meager”, $\mathcal{N}$ is “null”. Going up or to the right means the cardinal gets bigger and ZFC knows it. Each arrow can be made strict in a suitable model of ZFC. BJ refers to Bartoszyński/Judah.

Uniform transfer to the setting of computability

Most of this was described in Rupprecht's thesis in a more informal way. Full detail in Brooks-T, Brendle, Ng, Nies (BBNN) paper. Recall:

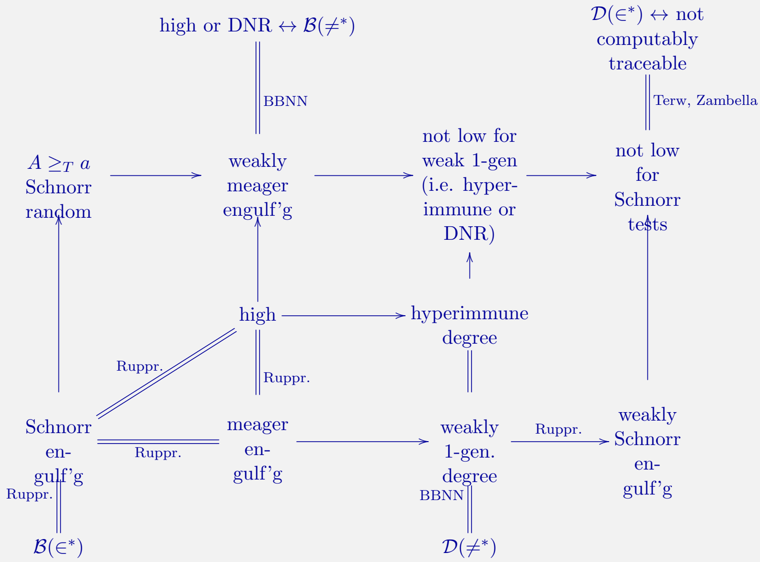
$$\begin{aligned}\mathfrak{b}(R) &= \min\{|F| : F \subseteq X \wedge \forall y \in Y \exists x \in F \neg xRy\} \\ \mathfrak{d}(R) &= \min\{|G| : G \subseteq Y \wedge \forall x \in X \exists y \in G\, xRy\}.\end{aligned}$$

Suppose we have specified what it means for objects x in X, y in Y to be computable in a Turing oracle A. Let the variable x range over X, and let y range over Y. We define the highness properties

$$\begin{aligned}\mathcal{B}(R) &= \{A : \exists y \leq_{\mathrm{T}} A\, \forall x \text{ computable } [xRy]\} \\ \mathcal{D}(R) &= \{A : \exists x \leq_{\mathrm{T}} A\, \forall y \text{ computable } [\neg xRy]\}.\end{aligned}$$

Note we are negating the set theoretic definitions. Reason: to "increase" a cardinal of the form $\min\{|F| \colon \phi(F)\}$, we need to introduce via forcing objects y so that $\phi(F)$ no longer holds in an extension model. This forcing corresponding to the construction of a powerful oracle computing a witness for $\neg\phi$.

Analog of Cichoń's diagram in computability theory (7 nodes)



Analog of non($\mathcal{N}$): weakly Schnorr engulfing

non($\mathcal{N}$) is the least size of a non-null set. (This is $\mathfrak{b}(\in_{\mathcal{N}})$, where $\in_{\mathcal{N}}$ is the element relation between reals and null sets.)

Determine the analogous highness property in computability.

- A Schnorr test is an effective sequence $(S_m)_{m\in\mathbb{N}}$ of Σ^0_1 sets in ${}^{\omega}2$ such that each λS_m is a computable real uniformly in m, and $\lambda S_m \leq 2^{-m}$. (λ is product measure on ${}^{\omega}2$.)
- A set $\mathcal{F} \subseteq {}^{\omega}2$ is Schnorr null if $\mathcal{F} \subseteq \bigcap_m S_m$ for a Schnorr test $(S_m)_{m\in\mathbb{N}}$.
- Each Schnorr null set fails to contain a computable set.

We say that A is weakly Schnorr engulfing (w.S.e.) if A computes a Schnorr test containing all computable reals. (This is the analog $\mathcal{B}(\in_{\mathcal{N}})$ of $\mathfrak{b}(\in_{\mathcal{N}})$. Introduced by Rupprecht.)

Known examples of A such that $\Gamma(A) \geq 1/2$

- The two known properties of A implying $\Gamma(A) \geq 1/2$ were:
  (1) Computably dominated random, and
  (2) computably traceable (= low for Schnorr null sets: every A-Schnorr null set is contained in a plain Schnorr null set).
- Both imply non-weakly Schnorr engulfing.
  (1) was proved by Rupprecht.
  (2) is trivial viewing the property as lowness for Schnorr null sets.
- Recent result of Kjos-Hanssen, Stephan and Terwijn: there is a non-w.S.e. without any of these properties. In fact it is non-DNR, and not computably traceable.

So let's show A non-w.S.e. implies that $\Gamma(A) \geq 1/2$. This will give a new type of example for such sets.

Theorem

Let A be not weakly Schnorr engulfing. Then $\Gamma(A) \geq 1/2$.

Proof. Let $B \leq_{\mathrm{T}} A$. Have to show $\gamma(B) \geq 1/2$.

- For each $d \in \mathbb{N}$ define a Schnorr test $(S_m)_{m \in \mathbb{N}}$ relative to A such that, for each set R not captured by this test, we have $\underline{\rho}(B \leftrightarrow R) \geq 1/2 - 1/d$.
- For each d, some computable set R passes the test. So this will show that $\gamma(B) \geq 1/2$.

Let $I_0 = \emptyset, I_1 = \{1\}, I_2 = \{2, 3\}, \ldots$.

- Given k, let $G_k =$ $\{Z \colon Z(i) \neq B(i)$ for a ratio of bits in I_k of at least $1/2 + 1/d\}$.
- G_k is a clopen set computed uniformly in k from A.
- Chernoff bounds: $\lambda G_k \leq e^{-2k/d^2}$.
- $S_m = \bigcup_{k \geq md^2} G_k$ defines a Schnorr test relative to A as required.

Characterization of w.S.e. via traces

An obvious question is whether conversely, $\Gamma(A) \geq 1/2$ implies that A is not weakly Schnorr engulfing. We studied w.S.e. in the hope of getting somewhere near.

Let $H : \omega \mapsto \omega$ be computable with $\sum 1/H(n)$ finite.
$\{T_n\}_{n\in\omega}$ is a *small computable H-trace* if

- T_n is a uniformly computable finite set
- $\sum_n |T_n|/H(n)$ is finite and computable.

The following is an analog of the (partial) combinatorial characterisation of $\mathrm{non}(\mathcal{N})$ in B/Judah book, Thm 2.5.15. (They need the extra hypothesis that $\mathrm{non}(\mathcal{N}) > \mathfrak{d}(\leq^*)$. No such thing is needed in the computability analog.)

Theorem

A is weakly Schnorr engulfing iff for some computable function H, there is an A-computable small H-trace capturing every computable function bounded by H.

Cichoń's diagram, again

$$
\begin{tikzcd}
 & \mathfrak{b}(\neq^*) \arrow[d, equal, "\text{BJ, Th. 2.4.7}"] & & \mathfrak{d}(\in^*) \arrow[d, equal, "\text{BJ, Th. 2.3.9}"] \\
\operatorname{cover}(\mathcal{N}) \arrow[r] & \operatorname{non}(\mathcal{M}) \arrow[r] & \operatorname{cofin}(\mathcal{M}) \arrow[r] & \operatorname{cofin}(\mathcal{N}) \\
 & \mathfrak{b}(\leq^*) \arrow[u] \arrow[r] & \mathfrak{d}(\leq^*) \arrow[u] & \\
\operatorname{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \operatorname{add}(\mathcal{M}) \arrow[u] \arrow[r] & \operatorname{cover}(\mathcal{M}) \arrow[u] \arrow[r] & \operatorname{non}(\mathcal{N}) \arrow[uu] \\
\mathfrak{b}(\in^*) \arrow[u, equal, "\text{BJ, Th. 2.3.9}"] & & \mathfrak{d}(\neq^*) \arrow[u, equal, "\text{BJ, Th. 2.4.1}"] &
\end{tikzcd}
$$

Analog of cover($\mathcal{M}$): infinitely often equal

cover($\mathcal{M}$) is the least size of a collection of meager sets with union $\mathbb{R}$. This coincides with $\mathfrak{d}(\neq^*)$, the least size of a set of functions in ω^ω such that for each function, some function in the set is a.e. different. We now use this to develop a new example of $\Gamma(A) = 0$.

We say that A is infinitely often equal (i.o.e.) if there is $g \leq_{\mathrm{T}} A$ such that $\exists^\infty n f(n) = g(n)$ for each computable function f.

This is easily seen to be equivalent to "A not computably dominated". And we already know this implies $\Gamma(A) = 0$. So what? Weaken it.

Let $H\colon \omega \to \omega$. We say that A is H-*infinitely often equal* if there is $g \leq_{\mathrm{T}} A$ such that $\exists^\infty n f(n) = g(n)$ for each computable function f bounded by H.

(This appears to get harder for A as H grows faster. If $H \geq 2$ is constant, H-i.o.e is the same as non-computable. However, we don't know that there is a proper hierarchy for functions H with $\infty > \sum_n 1/H(n)$.)

New example of $\Gamma(A) = 0$

Let $H\colon \omega \to \omega$. We say that A is H-infinitely often equal if there is $g \leq_{\mathrm{T}} A$ such that $\exists^\infty n f(n) = g(n)$ for each computable function f bounded by H.

Theorem

Let A be $2^{(\alpha^n)}$-i.o.e. for some $\alpha > 1$. Then $\Gamma(A) = 0$.

Previously known examples of sets A with $\Gamma(A) = 0$:

- not computably dominated, and
- degree of a completion of Peano arithmetic (PA for short).

Each property implies H-infinitely often equal for any given computable H.

Using a construction of Rupprecht (2010), given a computable $H \geq 2$, we can build an H-i.o.e. set A that is computably dominated, and not PA. So we have a new example of $\Gamma(A) = 0$.

New example of $\Gamma(A) = 0$

Theorem (again)

Let A be $2^{(\alpha^n)}$-i.o.e. for some computable $\alpha > 1$. Then $\Gamma(A) = 0$.

First we prove that A is 2^{k^n}-i.o.e. for any $k \in \mathbb{N}$.

Then we show that A 2^{k^n}-i.o.e. implies $\gamma(A) \leq 1/k$.

A cardinal characteristic corresponding to $\Gamma(A) \leq p$

This is somewhat speculative recent work with J. Brendle.

Let $0 \leq p < 1$. How many sets $x \subseteq \omega$ does one need so that: each set y is asymptotically equal to some x on a fraction of more than p bit positions?

$$\kappa_p = \min\{|F| \colon F \subseteq 2^\omega \wedge \forall y \in 2^\omega \exists x \in F \, \underline{\rho}(y \leftrightarrow x) > p\}.$$

Fact

(i) $p \leq q \Rightarrow \kappa_p \leq \kappa_q$.
(ii) $1/2 < p \leq q \Rightarrow \kappa_p = \kappa_q$.

(i) is trivial.
(ii) uses the "stretching" argument employed earlier on.

A set theoretic analog of the Γ question

$$\kappa_p = \min\{|F| \colon F \subseteq 2^\omega \wedge \forall y \in 2^\omega \exists x \in F\, \underline{\rho}(y \leftrightarrow x) > p\}.$$

For $p < 1/2$, we have

$$\text{cover}(\mathcal{M}) \leq \kappa_p \leq \text{non}(\mathcal{N}),$$

via set theoretic versions of the computability proofs mentioned earlier on.

E.g. for $\kappa_p \leq \text{non}(\mathcal{N})$, let $|F| < \kappa_p$. Choose y such that $\underline{\rho}(y \leftrightarrow x) \leq p$ for each $x \in F$. The set of such x is null (for almost every x, we must have $\underline{\rho}(y \leftrightarrow x) = 1/2$).

Question (Set theoretic Γ question)

Is it consistent that $\kappa_p < \kappa_q$ for some $p < q$?

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