The Gamma question and cardinal characteristics

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Sets and Computations
Joint work with Benoit Monin
The Gamma question and its variants
Lowness paradigms

Given a set \( A \subseteq \omega \). How close is \( A \) to being computable?

Several paradigms have been suggested.

- \( A \) has little power as a Turing oracle.
  E.g. computably dominated: every function \( g \leq_T A \) is dominated by a computable function.

- Many oracles compute \( A \).
  E.g. base for randomness: there is an oracle \( Z \), random in \( A \), such that \( Z \geq_T A \).

A recent paradigm: \( A \) is coarsely computable. This means there is a computable set \( R \) such that the asymptotic density of \( A \leftrightarrow R \) equals 1. Here \( A \leftrightarrow R := \{ n : A(n) = R(n) \} \).

The $\gamma$-value of a set $A \subseteq \omega$

The lower density of $Z \subseteq \omega$ is a number in $[0, 1]$, namely

$$\rho(Z) = \liminf_n \frac{|Z \cap [0, n]|}{n}$$

For the $\gamma$ value, smaller means harder to compute:

$$\gamma(A) = \sup_{R \text{ computable}} \rho(A \leftrightarrow R).$$

- If $\gamma(A) \geq p$ one says that $A$ is coarsely computable at density $p$.

- $A$ coarsely computable $\Rightarrow \gamma(A) = 1$.

Hirschfeldt et al. show that the converse fails.

Possible $\gamma$ values

- A computable: 1.
- A random: $1/2$

Any computable (in fact, left-$\Sigma^0_2$) real in $[0, 1]$ is a $\gamma(A)$.

Fact

Let $A$ be not computably dominated, i.e.

\[ \exists g \leq_T A \text{ not dominated by any computable function } h. \]

There is $B \leq_T A$ such that $\gamma(B) = 0$.

Proof idea: for $n > 0$ let $I_n = [(n-1)!, n!]$.

- For a computable set $R$ given by characteristic function $\Phi_e$, let $h(n)$ be the time it takes $\Phi_e$ to converge on $I_n$.
- If $h(n) < g(n)$, $A$ sees this convergence in time, and can make $B$ different from $R$ on all of $I_n$. This ensures

\[ \frac{|(B \leftrightarrow R) \cap [0, n!]|}{n!} \leq 1/n. \]

- Do some bookkeeping to treat all the total $\Phi_e$.

For such $e$ there are infinitely many $n$ with $h(n) < g(n)$.
Andrews et al. (2013) wanted to look at degrees, rather than sets. So they defined

\[ \Gamma(A) = \inf \{ \gamma(B) : B \equiv_T A \}. \]

One can as well take the inf over all \( B \leq_T A \).

- By previous fact, if \( A \) is not computably dominated then \( \Gamma(A) = 0 \).
- They show: if \( A \) is random and computably dominated, then \( \Gamma(A) = 1/2 \).

How about $\Gamma$ values in $(1/2, 1)$?

**Fact (Hirschfeldt et al.)**

*If $\Gamma(A) > 1/2$ then $A$ is computable (so that $\Gamma(A) = 1$).*

Proof idea: Again let $I_n = [(n - 1)!, n!]$ for $n > 0$.

- Define $B(k) = A(n)$ for $k \in I_n$. Then $B \equiv_T A$.
- Suppose that $\gamma(R \leftrightarrow B) > 1/2$ for a computable $R$.
- Then for almost all $n$
  (namely, all $n$ with $1/n < \Gamma(A) - 1/2$),
  $A(n)$ is the value of the majority of bits $R(k)$ for $k \in I_n$.
- This we can compute.
We have the $\Gamma$ values $1/2$, 1 and nothing properly in between. We also have the value 0. Is there any value properly in between 0 and 1?

**Question (\(\Gamma\)-question)**

Is there an $A$ such that $0 < \Gamma(A) < 1/2$?

- We won’t answer this.
- Rather, we use analogs of notions from cardinal characteristics to obtain natural classes of oracles with $\Gamma$ value $1/2$, and with $\Gamma$ value 0.
- This yields new examples for both cases.
- It may help to solve the problem eventually, by providing methods and a conceptual framework.
Variants of $\gamma$ and $\Gamma$

Other bases: We can work in base $b > 2$ rather than 2, i.e. $A: \omega \to \{0, \ldots, b - 1\}$.
$\gamma_b(A)$ and $\Gamma_b(A)$ are defined as expected.
We have $\Gamma_b(A) > 1/b \Rightarrow A$ computable (now with a harder proof). Values in $(0, 1/b)$?

Complexity theory: Fix an alphabet $\Sigma$. For $Z, A \subseteq \Sigma^*$ let

$$\underline{\rho}(Z) = \liminf \frac{|Z \cap \Sigma \leq n|}{|\Sigma \leq n|}$$

$$\gamma_{\text{poly}}(A) = \sup_{R \text{ poly time computable}} \underline{\rho}(A \leftrightarrow R)$$

$$\Gamma_{\text{poly}}(A) = \inf \{\gamma_{\text{poly}}(B) : B \equiv^p_T A\}.$$ 

Not much known here. Basic facts from computability don’t carry over. Which $\Gamma_{\text{poly}}$ values exist?
Cardinal characteristics
Cardinal characteristics and highness properties

Recent interaction of set theory and computability:
A close analogy between cardinal characteristics of the continuum, and highness properties (indicating strength of a Turing oracle).

- The domination number $\mathfrak{d}$ is the least size of a set of functions on the natural numbers so that every function is dominated by one of them.

- This corresponds to being not computably dominated: the oracle $A$ computes a function that is not dominated by any computable function.

- This correspondence was first studied explicitly by Nicholas Rupprecht, a student of A. Blass (thesis, article in Arch. of Math. Logic, 2010).
Domination, and slaloms (Bartoszyński, 1987)

- For \( f, g \in \omega \omega \), let \( f \leq^* g \iff f(n) \leq g(n) \) for almost all \( n \).
- A slalom is a function \( \sigma \) from \( \omega \) the finite subsets of \( \omega \) such that
  \[
  \forall n \ |\sigma(n)| \leq n^2.
  \]
- It traces \( f \) if
  \[
  \forall \infty n \ f(n) \in \sigma(n).
  \]
Set theory versus computability

\[ \varnothing = \varnothing(\leq^*) \] \quad \text{the least size of a set } F \subseteq \omega \omega \text{ dominating each function.}

\[ \text{cofin}(\mathcal{N}) \] \quad \text{the least size of a collection of null sets covering all null sets.}

\[ \varnothing(\in^*) \] \quad \text{the least size of a set of slaloms tracing all functions.}

\textbf{Thm.} [Bartoszyński 1984] \quad \varnothing(\in^*) = \text{cofin}(\mathcal{N}).

\[ \exists g \leq_T A \text{ such that } g \nleq^* f \text{ for each computable } f. \]

\[ A \text{ is not low for Schnorr tests.} \]

\[ A \text{ is not computably traceable.} \]

\textbf{Thm.} [Terw./Zamb. 2001] \quad \text{Computably traceable } \Leftrightarrow \text{ low for Schnorr tests}
Let $R \subseteq X \times Y$ be a relation. Let

\[
\begin{align*}
\mathfrak{b}(R) &= \min\{|F| : F \subseteq X \land \forall y \in Y \exists x \in F \neg x Ry\} \\
\mathfrak{d}(R) &= \min\{|G| : G \subseteq Y \land \forall x \in X \exists y \in G \ xRy\}.
\end{align*}
\]

- $\mathfrak{b}(R)$ is called the unbounding number of $R$, and $\mathfrak{d}(R)$ the domination number.
- If $R$ is a preordering without greatest element, then any set of covers is unbounded. So ZFC proves $\mathfrak{b}(R) \leq \mathfrak{d}(R)$. 
Extended Cichoń diagram of cardinals (10 nodes)

\[ \begin{array}{cccccc}
\text{cover}(\mathcal{N}) & \rightarrow & \text{non}(\mathcal{M}) & \rightarrow & \text{cofin}(\mathcal{M}) & \rightarrow & \text{cofin}(\mathcal{N}) \\
\text{add}(\mathcal{N}) & \rightarrow & \text{add}(\mathcal{M}) & \rightarrow & \text{cover}(\mathcal{M}) & \rightarrow & \text{non}(\mathcal{N})
\end{array} \]

\[ \begin{array}{cccccc}
b(\not=^*) & \quad & \mathfrak{d}(\in^*) & \quad & \mathfrak{b}(\leq^*) & \quad & \mathfrak{d}(\leq^*) \\
b(\in^*) & \quad & \mathfrak{d}(\not=^*) & \quad & \mathfrak{b}(\not=^*) & \quad & \mathfrak{d}(\not=^*)
\end{array} \]

\( \mathcal{M} \) denotes “meager”, \( \mathcal{N} \) is “null”. Going up or to the right means the cardinal gets bigger and ZFC knows it. Each arrow can be made strict in a suitable model of ZFC. BJ refers to Bartoszyński/Judah.
Uniform transfer to the setting of computability

Most of this was described in Rupprecht’s thesis in a more informal way. Full detail in Brooks-T, Brendle, Ng, Nies (BBNN) paper. Recall:

\[ b(R) = \min\{|F| : F \subseteq X \land \forall y \in Y \exists x \in F \neg xRy \} \]

\[ d(R) = \min\{|G| : G \subseteq Y \land \forall x \in X \exists y \in G xRy \} \].

Suppose we have specified what it means for objects \( x \) in \( X \), \( y \) in \( Y \) to be computable in a Turing oracle \( A \). Let the variable \( x \) range over \( X \), and let \( y \) range over \( Y \). We define the highness properties

\[ \mathcal{B}(R) = \{ A : \exists y \leq_T A \forall x \text{ computable } [xRy] \} \]

\[ \mathcal{D}(R) = \{ A : \exists x \leq_T A \forall y \text{ computable } [\neg xRy] \} \].

Note we are negating the set theoretic definitions. Reason: to “increase” a cardinal of the form \( \min\{|F| : \phi(F)\} \), we need to introduce via forcing objects \( y \) so that \( \phi(F) \) no longer holds in an extension model. This forcing corresponding to the construction of a powerful oracle computing a witness for \( \neg \phi \).
Analog of Cichoń’s diagram in computability theory (7 nodes)

- $A \geq_T a$
- Schnorr random
- $B(\notin^*)$
- Ruppr.
- $\mathcal{B}(\notin^*)$

- high or DNR $\leftrightarrow \mathcal{B}(\notin^*)$

- Not low for weak 1-gen (i.e. hyper-immune or DNR)

- $\mathcal{D}(\notin^*)$ $\leftrightarrow$ not computably traceable
- Terw, Zambella

- high
- hyperimmune degree
- weakly 1-gen. degree
- weakly Schnorr engulf’g

- weakly meager engulf’g
- not low for Schnorr tests

- $D(\notin^*)$
- Ruppr.
- Ruppr.
- Ruppr.
- Ruppr.
Analog of non($\mathcal{N}$): weakly Schnorr engulfing

non($\mathcal{N}$) is the least size of a non-null set. (This is $b(\in_{\mathcal{N}})$, where $\in_{\mathcal{N}}$ is the element relation between reals and null sets.)

Determine the analogous highness property in computability.

- A Schnorr test is an effective sequence $(S_m)_{m \in \mathbb{N}}$ of $\Sigma^0_1$ sets in $\omega^2$ such that each $\lambda S_m$ is a computable real uniformly in $m$, and $\lambda S_m \leq 2^{-m}$. ($\lambda$ is product measure on $\omega^2$.)
- A set $\mathcal{F} \subseteq \omega^2$ is Schnorr null if $\mathcal{F} \subseteq \bigcap_m S_m$ for a Schnorr test $(S_m)_{m \in \mathbb{N}}$.
- Each Schnorr null set fails to contain a computable set.

We say that $A$ is weakly Schnorr engulfing (w.S.e.) if $A$ computes a Schnorr test containing all computable reals. (This is the analog $B(\in_{\mathcal{N}})$ of $b(\in_{\mathcal{N}})$. Introduced by Rupprecht.)
Known examples of $A$ such that $\Gamma(A) \geq 1/2$

- The two known properties of $A$ implying $\Gamma(A) \geq 1/2$ were:
  1. Computably dominated random, and
  2. computably traceable (= low for Schnorr null sets: every $A$-Schnorr null set is contained in a plain Schnorr null set).

- Both imply non-weakly Schnorr engulfing.
  1. was proved by Rupprecht.
  2. is trivial viewing the property as lowness for Schnorr null sets.

- Recent result of Kjos-Hanssen, Stephan and Terwijn: there is a non-w.S.e. without any of these properties. In fact it is non-DNR, and not computably traceable.

So let’s show $A$ non-w.S.e. implies that $\Gamma(A) \geq 1/2$. This will give a new type of example for such sets.
Theorem

Let $A$ be not weakly Schnorr engulfing. Then $\Gamma(A) \geq 1/2$.

Proof. Let $B \leq_T A$. Have to show $\gamma(B) \geq 1/2$.

- For each $d \in \mathbb{N}$ define a Schnorr test $(S_m)_{m \in \mathbb{N}}$ relative to $A$ such that, for each set $R$ not captured by this test, we have $\rho(B \leftrightarrow R) \geq 1/2 - 1/d$.

- For each $d$, some computable set $R$ passes the test. So this will show that $\gamma(B) \geq 1/2$.

Let $I_0 = \emptyset, I_1 = \{1\}, I_2 = \{2, 3\}, \ldots$.

- Given $k$, let $G_k = \{Z : Z(i) \neq B(i) \text{ for a ratio of bits in } I_k \text{ of at least } 1/2 + 1/d\}$.

- $G_k$ is a clopen set computed uniformly in $k$ from $A$.

- Chernoff bounds: $\lambda G_k \leq e^{-2k/d^2}$.

- $S_m = \bigcup_{k \geq md^2} G_k$ defines a Schnorr test relative to $A$ as required.
Characterization of w.S.e. via traces

An obvious question is whether conversely, $\Gamma(A) \geq 1/2$ implies that $A$ is not weakly Schnorr engulfing. We studied w.S.e. in the hope of getting somewhere near.

Let $H : \omega \mapsto \omega$ be computable with $\sum 1/H(n)$ finite. 

$\{T_n\}_{n \in \omega}$ is a small computable $H$-trace if 

- $T_n$ is a uniformly computable finite set
- $\sum_n |T_n|/H(n)$ is finite and computable.

The following is an analog of the (partial) combinatorial characterisation of non$(\mathcal{N})$ in B/Judah book, Thm 2.5.15.

(They need the extra hypothesis that non$(\mathcal{N}) > \omega(\leq^*)$. No such thing is needed in the computability analog.)

**Theorem**

A is weakly Schnorr engulfing iff for some computable function $H$, there is an $A$-computable small $H$-trace capturing every computable function bounded by $H$. 
Cichoń’s diagram, again

\[ \text{cover}(\mathcal{N}) \rightarrow \text{non}(\mathcal{M}) \rightarrow \text{cofin}(\mathcal{M}) \rightarrow \text{cofin}(\mathcal{N}) \]

\[ \text{add}(\mathcal{N}) \rightarrow \text{add}(\mathcal{M}) \rightarrow \text{cover}(\mathcal{M}) \rightarrow \text{non}(\mathcal{N}) \]

\[ b(\neq^*) \]

\[ d(\in^*) \]

BJ, Th. 2.4.7 \hspace{1cm} BJ, Th. 2.3.9

BJ, Th. 2.3.9 \hspace{1cm} BJ, Th. 2.4.1
Analog of $\text{cover}(\mathcal{M})$: infinitely often equal

$\text{cover}(\mathcal{M})$ is the least size of a collection of meager sets with union $\mathbb{R}$. This coincides with $\mathfrak{d}(\neq^*)$, the least size of a set of functions in $\omega^\omega$ such that for each function, some function in the set is a.e. different. We now use this to develop a new example of $\Gamma(A) = 0$.

We say that $A$ is infinitely often equal (i.o.e.) if there is $g \leq_T A$ such that $\exists \infty n f(n) = g(n)$ for each computable function $f$.

This is easily seen to be equivalent to “$A$ not computably dominated”. And we already know this implies $\Gamma(A) = 0$. So what? Weaken it.

Let $H : \omega \rightarrow \omega$. We say that $A$ is $H$-infinitely often equal if there is $g \leq_T A$ such that $\exists \infty n f(n) = g(n)$ for each computable function $f$ bounded by $H$.

(This appears to get harder for $A$ as $H$ grows faster. If $H \geq 2$ is constant, $H$-i.o.e is the same as non-computable. However, we don’t know that there is a proper hierarchy for functions $H$ with $\infty > \sum_n 1/H(n)$.)
New example of $\Gamma(A) = 0$

Let $H : \omega \to \omega$. We say that $A$ is $H$-infinitely often equal if there is $g \leq_T A$ such that $\exists \infty n f(n) = g(n)$ for each computable function $f$ bounded by $H$.

Theorem

Let $A$ be $2^{(\alpha^n)}$-i.o.e. for some $\alpha > 1$. Then $\Gamma(A) = 0$.

Previously known examples of sets $A$ with $\Gamma(A) = 0$:

▶ not computably dominated, and
▶ degree of a completion of Peano arithmetic (PA for short).

Each property implies $H$-infinitely often equal for any given computable $H$.

Using a construction of Rupprecht (2010), given a computable $H \geq 2$, we can build an $H$-i.o.e. set $A$ that is computably dominated, and not PA. So we have a new example of $\Gamma(A) = 0$. 
New example of $\Gamma(A) = 0$

Theorem (again)

Let $A$ be $2^{(\alpha^n)}$-i.o.e. for some computable $\alpha > 1$. Then $\Gamma(A) = 0$.

First we prove that $A$ is $2^{k^n}$-i.o.e. for any $k \in \mathbb{N}$.

Then we show that $A$ $2^{k^n}$-i.o.e. implies $\gamma(A) \leq 1/k$. 
A cardinal characteristic corresponding to $\Gamma(A) \leq p$

This is somewhat speculative recent work with J. Brendle.

Let $0 \leq p < 1$. How many sets $x \subseteq \omega$ does one need so that: each set $y$ is asymptotically equal to some $x$ on a fraction of more than $p$ bit positions?

$$\kappa_p = \min\{|F| : F \subseteq 2^\omega \land \forall y \in 2^\omega \exists x \in F \rho(y \leftrightarrow x) > p\}.$$  

Fact

(i) $p \leq q \Rightarrow \kappa_p \leq \kappa_q$.
(ii) $1/2 < p \leq q \Rightarrow \kappa_p = \kappa_q$.

(i) is trivial.
(ii) uses the “stretching” argument employed earlier on.
A set theoretic analog of the $\Gamma$ question

$$\kappa_p = \min\{|F| : F \subseteq 2^\omega \land \forall y \in 2^\omega \exists x \in F \rho(y \leftrightarrow x) > p\}.$$ 

For $p < 1/2$, we have

$$\text{cover}(\mathcal{M}) \leq \kappa_p \leq \text{non}(\mathcal{N}),$$

via set theoretic versions of the computability proofs mentioned earlier on.

E.g. for $\kappa_p \leq \text{non}(\mathcal{N})$, let $|F| < \kappa_p$. Choose $y$ such that $\rho(y \leftrightarrow x) \leq p$ for each $x \in F$. The set of such $x$ is null (for almost every $x$, we must have $\rho(y \leftrightarrow x) = 1/2$).

**Question (Set theoretic $\Gamma$ question)**

Is it consistent that $\kappa_p < \kappa_q$ for some $p < q$?
References


- These slides on my/the IMS web page.