Local compactness for computable Polish metric spaces is Π^1_1-complete

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Goal

We are given a class $\mathcal{C}$ of structures: countable abelian groups, countable Boolean algebras, Polish metric spaces,...

We consider isomorphism invariant properties of structures in $\mathcal{C}$, and similarity relations between objects in $\mathcal{C}$.

We study the complexity of such properties and relations. Often this is determined by being complete in its class of descriptive complexity.

Two settings:

Descriptive set theory: represent structure by a real.

Computability theory: only look at computable structures in $\mathcal{C}$. Represent such a structure by a computable index.

Example: complexity of being torsion

A computable abelian group is given by a binary function ϕ_e on $\mathbb{N}$ describing $+$, and a number denoting the neutral element.
It is Π^0_2 to say that for such a representation, ϕ_e is total and it describes an abelian group. Being torsion is also Π^0_2.

Fact

Being torsion is Π^0_2 complete for computable abelian groups.

Proof

- Reduce the Π^0_2 complete problem $\{i\colon\ W_i = \mathbb{N}\}$ to being torsion.
- From i, n, we can compute a representation of a group G_n that is cyclic of order $2s$ if n enters W_i at stage s, and infinite cyclic otherwise.
- Let H_i be a computable representation of $\bigoplus_n G_n$. Then $W_i = \mathbb{N} \Leftrightarrow H_i$ is torsion.

Similarity relations

Let E, F be equivalence relations on $\mathbb{N}$. Computable reducibility:

$$E \preceq_c F \text{ if } \forall u \forall v \, Euv \leftrightarrow Fh(u)h(v)$$

for a computable function $h\colon \mathbb{N} \to \mathbb{N}$.

Theorem (Friedman, Fokina, and N., 2012)

Computable isomorphism of computable Boolean algebras is complete for Σ^0_3 equivalence relations.

Theorem (follows from Camerlo/Gao 2000 and Friedman, Fokina +4)

Plain isomorphism of computable Boolean algebras is complete for Σ^1_1 equivalence relations.

FF+4 showed this for isomorphism of computable graphs. Camerlo/Gao had encoded countable graphs into Boolean algebras preserving isomorphism. Nies and Solecki observed that their construction is effective. In fact, Camerlo/Gao built Stone spaces. So we also have:

Homeomorphism of compact computable metric spaces is complete for Σ^1_1 equivalence relations.

Representing Polish metric spaces

Definition

- A Polish metric space $\mathcal{M}$ is a complete metric space (M, d) together with a dense sequence $(p_i)_{i \in \mathbb{N}}$.
- The space is computable if $d(p_i, p_k)$ is a computable real, uniformly in i, k.

- In descriptive set theory one sees the representations of Polish metric spaces as a G_δ set $\mathcal{P} \subseteq \mathbb{R}^{\omega \times \omega}$. This is sometimes called the hyperspace of Polish spaces.
- The computable spaces can be seen as the computable points in this hyperspace.

In computability theory, we represent computable Polish metric spaces by an index e for the distance function:

$\phi_e(i, k, n)$ is a rational approximating $d(p_i, p_k)$ up to error 2^{-n}.

Complexity of being compact

For complete metric spaces, compactness is equivalent to the existence of finite ϵ-nets for each rational $\epsilon > 0$:

$$\forall \epsilon > 0\, \exists n \forall r \bigvee_{i<n} d(p_i, p_r) \leq \epsilon.$$

This is Π^0_3.

Proposition (Melnikov and N., CiE 2013)

The set of computable indices e for compact metric spaces $\mathcal{M}_e$ is Π^0_3 complete.

They also showed that the complexity of isometry of compact spaces is merely Π^0_2 within that Π^0_3 set.

Complexity of being locally compact

Definition

A topological space is locally compact if every point has a compact neighbourhood.

- For computable Polish metric spaces, this property is Π^1_1:
- express that for every point x, there is a positive rational ϵ such that the closed ball of radius ϵ around x is compact.
- This is Π^0_3 in a Cauchy name for x by relativizing the condition for compactness above.

Theorem (N. and Solecki)

The set of computable indices for locally compact metric spaces is Π^1_1-complete.

Theorem (N. and Solecki)

The set of computable indices for locally compact metric spaces is Π^1_1-complete.

- Let $I_{\mathbb{Q}} = [0,1] \cap \mathbb{Q}$.
- By an index for a computable subset R of $I_{\mathbb{Q}}$ we mean a number e such that ϕ_e, interpreted as a function $I_{\mathbb{Q}} \to \mathbb{N}$, is the characteristic function of R. We write $R = R_e$.
- Let $O \subseteq \omega$ denote a Π^1_1 complete set.

Example of a compact subset of $I_{\mathbb{Q}}$: $\{1/2\} \cup \{1/2 - 2^{-n} \colon n > 0\}$.
Example of a closed, but non-compact subset of $I_{\mathbb{Q}}$: sequence of rationals converging to $\pi/4$.

By the following, the set of indices for compact computable subsets of $I_{\mathbb{Q}}$ is Π^1_1-complete. (Note that being closed for a set R_e is merely Π^0_3.)

Fact (by an ancient result of Hurewicz)

There is a computable function g such that $R_{g(e)}$ is closed in $I_{\mathbb{Q}}$ for each e (with $g(e)$ defined), and $e \in O \leftrightarrow R_{g(e)}$ is compact.

Proof

Show that $\{i\colon\ R_i \text{ is compact }\} \leq_m \{e\colon\ M_e \text{ locally compact}\}$

where M_e is the e-th computable metric space.

- Let $\langle q_k \rangle_{k\in\mathbb{N}}$ effectively list $I_{\mathbb{Q}}$ without repetition.
  Pick a computable metric $d_{\mathcal{N}}$ on Baire space $\mathcal{N} = [0,1] - \mathbb{Q}$.
- Given a closed $R = R_i$, let
$$\Theta(R) = [0,1] \cap \{q_k - 2^{-m}\sqrt{2}\colon\ q_k \in R, m \geq k\}.$$
- Effectively list these points as $\langle v_j \rangle_{j\in\mathbb{N}}$.
- Effectively obtain a computable index e for a distance matrix $d_{\mathcal{N}}(v_i, v_j)$, representing $M_e = \text{Closure}_{\mathcal{N}}(\Theta(R))$.

$$\text{Closure}_{[0,1]}(\Theta(R)) = R \,\dot{\cup}\, \text{Closure}_{\mathcal{N}}(\Theta(R)).$$

Claim

R is compact $\Leftrightarrow$ $\text{Closure}_{\mathcal{N}}(\Theta(R))$ is locally compact.

Use: if $\bar{Y} = X$, X compact, then Y is locally compact $\leftrightarrow$ Y is open.
Here X is the closure in $[0,1]$, and Y the closure in $\mathcal{N}$.

Example for this proof

- Example of compact subset of $I_{\mathbb{Q}}$:

  $$R = \{1/2\} \cup \{1/2 - 2^{-n}:\ n > 0\}.$$

  Then $\Theta(R)$ is discrete in $\mathcal{N}$, hence locally compact.
- Example of closed, but non-compact subset of $I_{\mathbb{Q}}$:

  $R =$ members of a sequence of rationals $\langle a_r \rangle$ converging to $\pi/4$.

  - Then $\mathrm{Closure}_{\mathcal{N}}(\Theta(R))$ contains $\pi/4$.
  - Every nbhd of $\pi/4$ contains a tail of a sequence $a_r - 2^{-m}\sqrt{2}$ converging to an a_r, but not a_r itself.
  - So this tail sequence has no convergent subsequence in $\mathrm{Closure}_{\mathcal{N}}(\Theta(R))$.

Ultrametric version

A metric space (M, d) is called ultrametric if

$$\forall x, y, z \in M\,[d(x, z) \leq \max\{d(x, y), d(y, z)\}].$$

- We can carry out the above proof in Cantor space $2^{\mathbb{N}}$, with the natural homeomorphic embedding of Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ as the sequences with infinitely many 1's.
- This shows that the set of computable ultrametric locally compact metric spaces is Π^1_1-complete.

Some directions and open questions

Question (Computability)

Determine the descriptive complexity of being connected among computable Polish metric spaces. (The trivial upper bound is Π^1_2.)

Question (Descriptive set theory)

Is an the complete orbit equivalence relation Borel reducible to isometry of locally compact Polish metric spaces?

Suggestions by T. Tsankov in descr. set theory:

- study the meaning of the Π^1_1-rank for the class of locally compact spaces.
- study the representation of locally compact m.s. as $K - \{x\}$, where K is a compact metric space and $x \in K$.