Randomness and analysis: a tutorial

Part II: Lebesgue density and its applications to randomness

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CCC 2015, Kochel am See

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Density

Let λ denote uniform (Lebesgue) measure.

Definition

Let E be a subset of $[0, 1]$. The *(lower) density* of E at a real z is

$$\underline{\rho}(E \mid z) = \liminf_{z \in J\,,\, |J| \to 0} \frac{\lambda(J \cap E)}{|J|},$$

where J ranges over intervals.

This gauges how much, at least, of E is in intervals that zoom in on z.

$\rho(E \mid z)$ is the limit over intervals containing z. Clearly

$$\underline{\rho}(E \mid z) = 1 \leftrightarrow \rho(E \mid z) = 1.$$

Lebesgue's Theorem: towards an effective version

Recall: $\underline{\rho}(E \mid z) = \liminf_{J \text{ interval}, z \in J, |J| \to 0} \lambda(J \cap E)/|J|$.

Theorem (Lebesgue Density Theorem, 1910)

Let $E \subseteq [0,1]$ be measurable. Then for almost every $z \in [0,1]$: if $z \in E$, then $\underline{\rho}(E \mid z) = 1$.

- For open E, this is immediate, and actually holds for all $z \in [0,1]$.
- For closed E, this is the simplest case where there is something to prove.

$E \subseteq [0,1]$ is effectively closed (or Π^0_1) if there is an effective list of open intervals with rational endpoints that has union $[0,1] \setminus E$.

Definition (Main)

We say that a real z is a density-one point if $\underline{\rho}(E \mid z) = 1$ for every effectively closed $E \ni z$.

Martin-Löf randomness and density

Does Martin-Löf randomness ensure that an effectively closed $E \subseteq [0,1]$ with $z \in E$ has density one at z?

Answer: NO!

Example

- Let $E \neq \emptyset$, $E \subseteq [0,1]$ be an effectively closed set containing only Martin-Löf randoms.
- E.g., $E = [0,1] \setminus S_1$ where $\langle S_r \rangle_{r \in \mathbb{N}}$ is a universal ML-test.
- Let $z = \min(E)$.
- Then $\underline{\rho}(E \mid z) = 0$ even though z is ML-random.

Density randomness

Definition

Let E be a measurable subset of $2^{\mathbb{N}}$. The *lower dyadic density* of E at $Z \in 2^{\mathbb{N}}$ is

$$\underline{\rho}_2(E \mid Z) = \liminf_{n \to \infty} 2^n \lambda([Z \restriction_n] \cap E).$$

Definition

We say that $Z \subseteq \mathbb{N}$ is density random if Z is ML-random and $\underline{\rho}_2(P \mid Z) = 1$ for each Π^0_1 class $P \in Z$.

For ML-random Z, one can equivalently require the full density equals 1 in the setting of reals, by a result of Khan and Miller (2013).

Three characterisations of density randomness

Theorem

The following are equivalent for $Z \in 2^{\mathbb{N}}$, $z = 0.Z$.

- Z is density random
- [Madison group, 2012] Each left-c.e. martingale M converges: $\lim_n M(Z\upharpoonright_n)$ exists (M is left-c.e. if $M(\sigma)$ is a left-c.e. real uniformly in string σ)
- [N., 2014] $g'(z)$ exists for each interval-c.e. function g
- [Miyabe, N., Zhang 2013] For each integrable lower semicomputable function $f\colon [0,1] \to \overline{\mathbb{R}}^+$, the "averaging" statement of the Lebesgue differentiation theorem holds at z.

For background and complete proofs see Miyabe, N., Zhang 2013. The continuous interval-c.e. functions with $g(0) = 0$ are precisely the variation functions of computable functions by Freer et al. 2014.

2. Anti-random sequences

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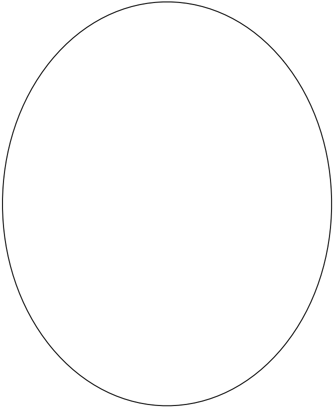
Basic objects of computability theory



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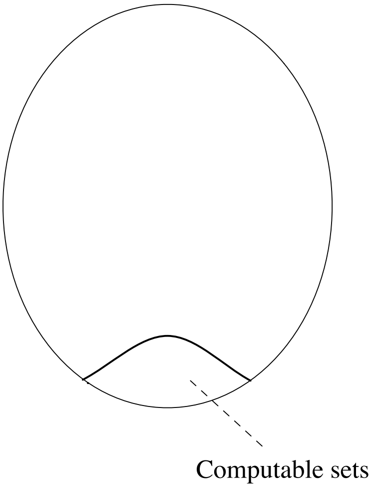

- The computable subsets of $\mathbb{N}$

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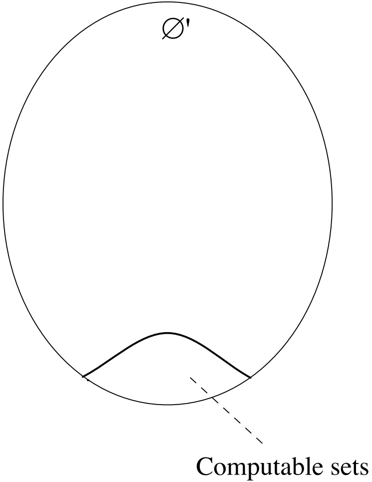

- The computable subsets of $\mathbb{N}$
- the halting problem $\emptyset'$

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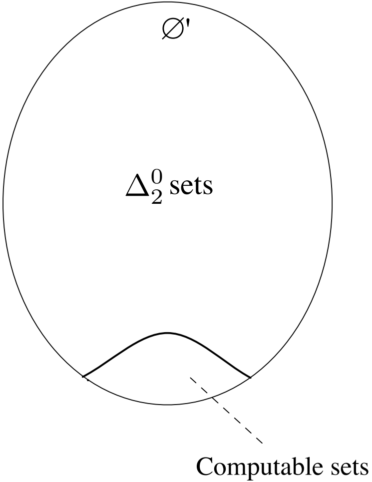

- The computable subsets of $\mathbb{N}$
- the halting problem $\emptyset'$
- Turing reducibility $\leq_T$
- the Δ^0_2 sets ($A \leq_{\mathrm{T}} \emptyset'$)

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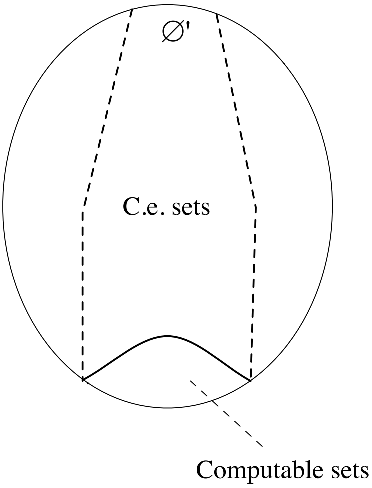

- The computable subsets of $\mathbb{N}$
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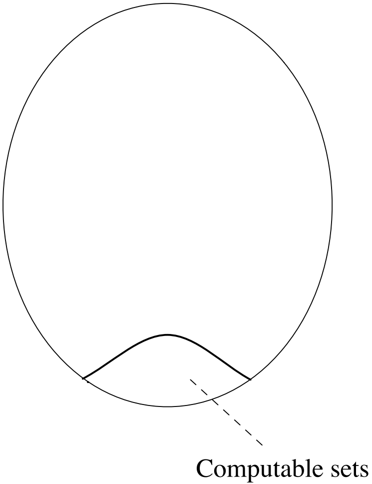
- Turing reducibility $\leq_T$
- the computably enumerable sets

Adding the world of (anti-)randomness



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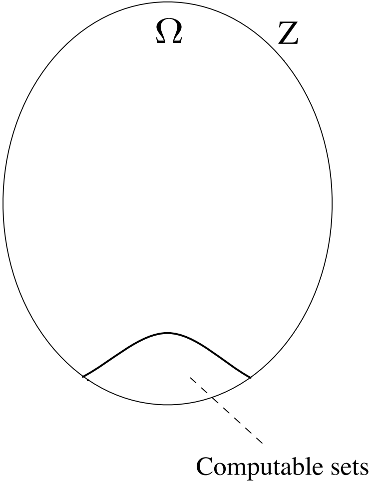

- The Martin-Löf random sets Z, such as Chaitin's halting probability Ω. We have $\Omega \equiv_T \emptyset'$.

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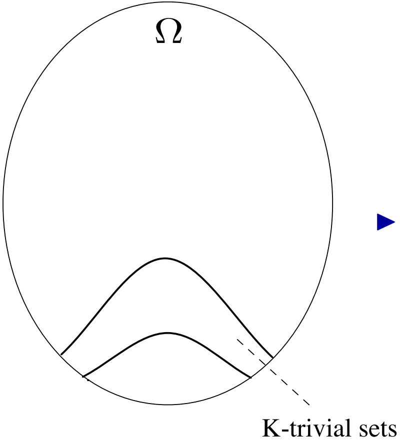

- The antirandom (K-trivial) sets.

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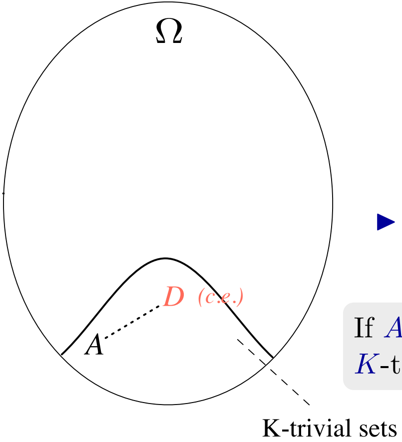
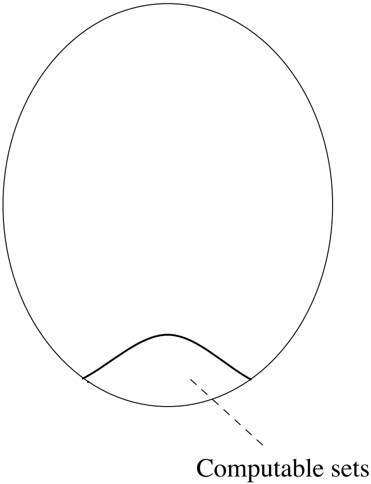


- The antirandom (K-trivial) sets.

If A is K-trivial, then there is c.e. K-trivial set $D \geq_{tt} A$. (N. 2005)

Kučera's theorem and the covering problem



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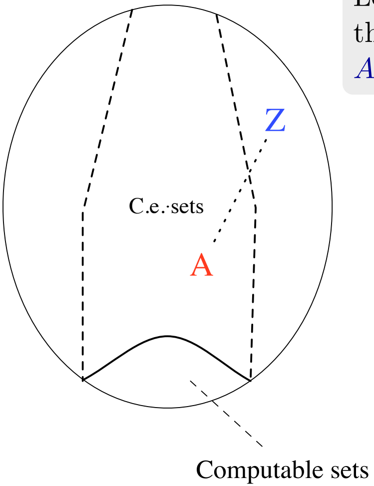
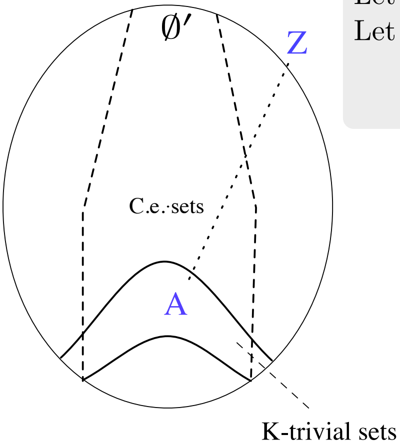


Let Z be a random Δ^0_2 set. Then there is a c.e., incomputable set $A \leq_T Z$. (Kučera, 1986)

Kučera's theorem and the covering problem



Let Z be random with $Z \not\geq_T \emptyset'$.
Let $A \leq_{\mathrm{T}} Z$ be c.e.

Then A is K-trivial.

(Hirschfeldt, N., Stephan, 2007)

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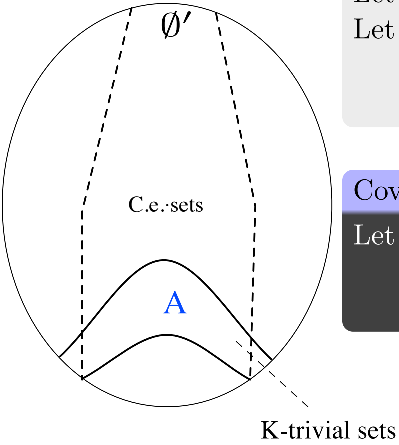

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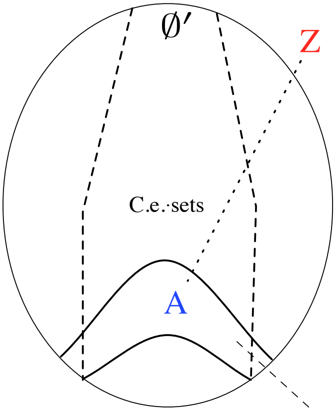

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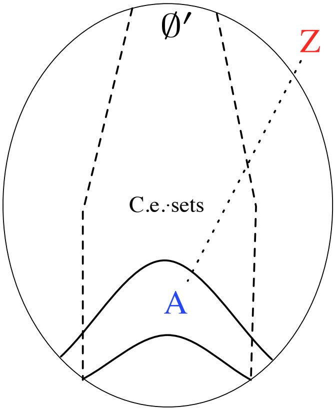

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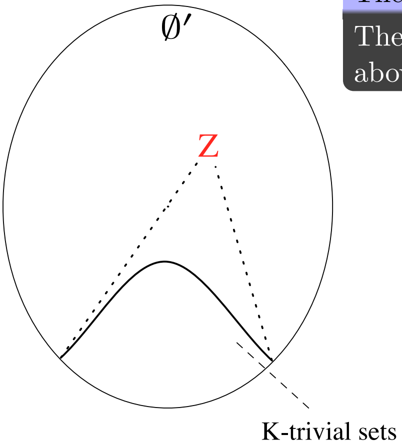
Let A be a c.e. K-trivial set.
Is there a ML-random $Z \geq_T A$ with $Z \not\geq_T \emptyset'$?

We may omit the assumption that A is c.e.: if not, replace A by a c.e. K-trivial set D above A.

A strong solution to the covering problem



Theorem 5 + 2 authors

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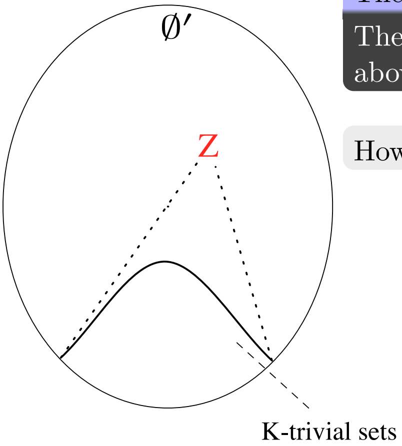

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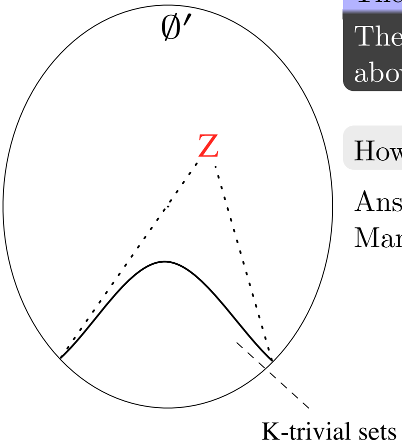

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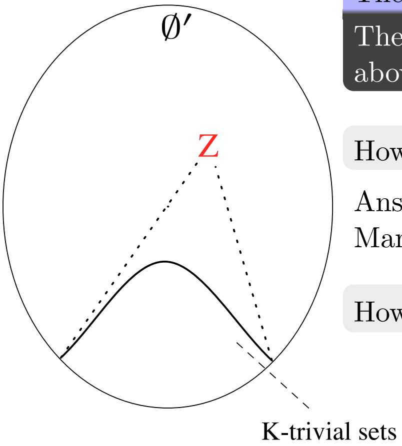

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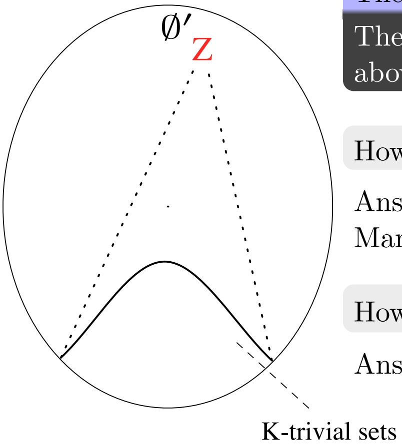

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Answer: not much more than Martin-Löf random.

How close to $\emptyset'$ must Z lie?

Answer: Z is very close to $\emptyset'$.

Background on antirandom sets

Descriptive string complexity K

Consider a partial computable function from binary strings to binary strings (called machine). It is called prefix-free if its domain is an antichain under the prefix relation of strings.

There is a universal prefix-free machine $\mathbb{U}$:
for every prefix-free machine M,

$$M(\sigma) = y \text{ implies } \mathbb{U}(\tau) = y \text{ for some } \tau \text{ with } |\tau| \leq |\sigma| + d_M,$$

and the constant d_M only depends on M.

The prefix-free Kolmogorov complexity of string y is the length of a shortest $\mathbb{U}$-description of y:

$$K(y) = \min\{|\sigma| : \mathbb{U}(\sigma) = y\}.$$

Definition of K-triviality

In the following, we identify a natural number n with its binary representation (as a string). For a string τ, up to additive const we have $K(|\tau|) \leq K(\tau)$, since we can compute $|\tau|$ from τ.

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Definition (going back to Chaitin, 1975)

An infinite sequence of bits A is K-trivial if, for some $b \in \mathbb{N}$,

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namely, all its initial segments have minimal K-complexity.

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$$\begin{array}{lllll} Z \text{ is} & \text{random} & \Leftrightarrow & \forall n \, [K(Z \upharpoonright_n) > n & -O(1)] \\ A \text{ is} & K\text{-trivial} & \Leftrightarrow & \forall n \, [K(A \upharpoonright_n) \leq K(n) & +O(1)] \end{array}$$

Thus, being K-trivial means being *far from random*.

Connecting density and K-triviality

This is based on the following work:

[Oberwolfach]	Bienvenu, Greenberg, Kučera, N. Turetsky JEMS, to appear	2012
[Berkeley]	Day and Miller Math. Research Letters, to appear	2012
[Paris]	Bienvenu, Miller, Hölzl and N. STACS 2012, JML 2014	2011

Turing incompleteness and positive density

Definition

We say that a real z is a positive density point if
$$\underline{\rho}(E \mid z) > 0 \text{ for every effectively closed } E \ni z.$$

For a real $z \notin \mathbb{Q}$, let $Z \in 2^{\mathbb{N}}$ denote its binary expansion: $z = 0.Z$.

Theorem (Paris)

Let z be a Martin-Löf random real. Then
Z is NOT above the halting problem $\emptyset' \Leftrightarrow$
z is a positive density point.

The main connection of density and K-trivials

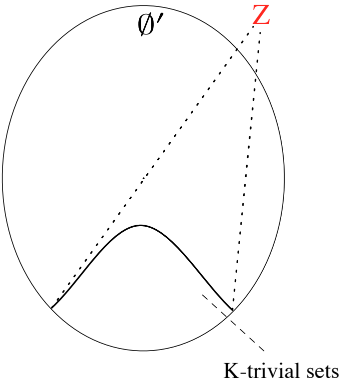
Recall: $\underline{\rho}(E \mid z) = \liminf_{|J| \to 0, z \in J} \lambda(J \cap E)/|J|$.

Definition (Recall)

We say that a real z is a density-one point if

$$\underline{\rho}(E \mid z) = 1 \text{ for every effectively closed } E \ni z.$$

In other words, z satisfies the Lebesgue Theorem for effectively closed sets.



Theorem (Oberwolfach)

Let z be a Martin-Löf random real. Suppose z is NOT a density-one point.

Then Z is above all the K-trivials.

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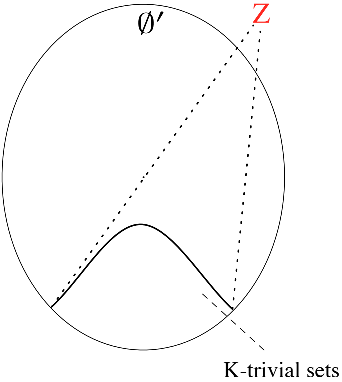

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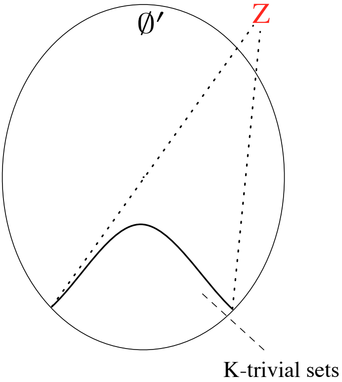

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To solve the covering problem, we need to know:

Does Z as in the picture exist?

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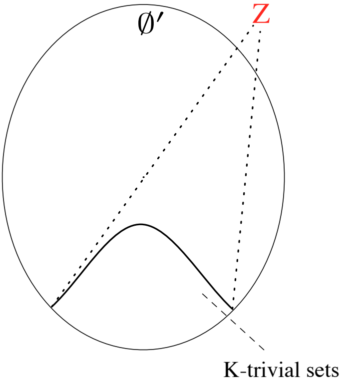

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To solve the covering problem, we need to know:

Does Z as in the picture exist?

Where do we get a ML-random set $Z \not\geq_T \emptyset'$ that is not a density-one point?

Why such a Z exists

Theorem (Berkeley, i.e., Day and Miller)

Let P be a nonempty Π^0_1 class of ML-randoms. There is a ML-random set $Z \not\geq_T \emptyset'$ such that $\underline{\rho}_2(P \mid Z) \leq 1/2$.

[Paris] characterized difference randomness of a ML-random Z via positive density:

$$Z \not\geq_T \emptyset' \text{ iff } Z \text{ is a positive density point.}$$

Berkeley built a set Z that is a positive density point.

Note that, for the Day-Miller set Z, the local measure $\lambda_\sigma(Z)$ for $\sigma \prec Z$ oscillates between 1 (asymptotically), and a value ϵ with $0 < \epsilon \leq 1/2$.

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Proof. Force with conditions of the form $\langle \sigma, Q \rangle$, where

- σ is a string, $Q \subseteq P$, $[\sigma] \cap Q \neq \emptyset$
- there is $\delta < 1/2$ such that each string $\tau \succeq \sigma$ has two options:

$$\text{either } [\tau] \cap Q = \emptyset, \text{ or } \lambda_\tau(Q) \geq \lambda_\tau(P) - \delta.$$

(Q either loses all, or $\leq \delta$ of P's local measure within $[\tau]$.)

$\langle \sigma', Q' \rangle$ extends $\langle \sigma, Q \rangle$ if $\sigma' \succeq \sigma$ and $Q' \subseteq Q$.
We have an initial condition $\langle \emptyset, P \rangle$ (via $\delta = 0$).

If G is a sufficiently generic filter, then $Z_G = \bigcup \{\sigma \colon \langle \sigma, Q \rangle \in G\}$ is a ML-random positive density point, and $\rho_2(P \mid Z) \leq 1/2$.
Then by Bienvenu et al., $Z \not\geq_T \emptyset'$.

The strongest answer to the covering question

Berkeley's careful effectvization of the forcing yields a Δ^0_2 set Z.

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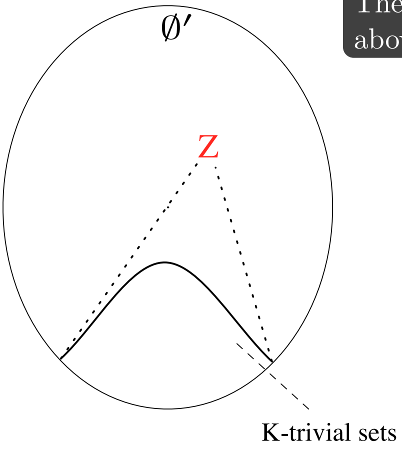


Theorem (Oberwolfach + Berkeley)

There is a ML-random set $Z <_T \emptyset'$ above all the K-trivials.

Questions on density randomness

Question (Turetsky)

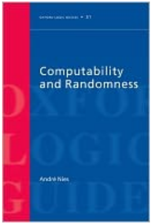
Is density randomness closed downward within the ML-randoms?

This is known for most randomness notions stronger than Martin-Löf's, including for OW-randomness (by the results above).

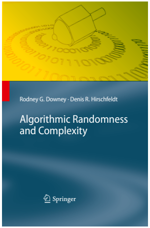
Question (Franklin)

Is density randomness equivalent to being a Birkhoff point for each computable measure preserving operator and semicomputable function?

Book references for background



My book
"Computability and Randomness",
Oxford University Press, 447 pages, Feb. 2009;
Paperback version Mar. 2012.



Book by Downey and Hirschfeldt:
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Springer, $>$ 800 pages, Dec. 2010;

Paper and preprint references for Part II

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