A unifying approach to the Gamma question

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Lowness paradigms

Given a set $A \subseteq \mathbb{N}$. How close is $A$ to being computable?

Several paradigms have been suggested and studied.

- $A$ has little power as a Turing oracle.
- Many oracles compute $A$.

A recent paradigm: $A$ is coarsely computable.
This means there is a computable set $R$ such that the asymptotic density of $\{n: A(n) = R(n)\}$ equals 1.

The $\gamma$-value of a set $A \subseteq \mathbb{N}$

A computable set $R$ tries to approximate a complicated set $A$:

$$A : 100100100100 000101001001 010101111010 101010100111$$
$$R : \underline{00001011}0111 \underline{01010100}0101 010001011010 101010100111$$

$\approx 1/2$ correct

$\approx 2/3$ correct

$\approx 3/4$ correct

$\approx 4/5$ correct

Take sup of the asymptotic correctness over all computable $R$'s:

$$\gamma(A) = \sup_{R \text{ computable}} \rho\{n : A(n) = R(n)\}$$

where $\rho(Z) = \lim \inf_n \frac{|Z \cap [0, n)|}{n}$. 
Some examples of values $\gamma(A)$

Recall

$$\gamma(A) = \sup_{R \text{ computable}} \rho\{n: A(n) = R(n)\}$$

where $\rho(Z) = \liminf_n \frac{|Z \cap [0, n]|}{n}$.

Some possible values

- $A$ computable $\Rightarrow \gamma(A) = 1$
- $A$ random $\Rightarrow \gamma(A) = 1/2$. 
Andrews, Cai, Diamondstone, Jockusch and Lempp (2013) looked at Turing degrees, rather than sets. They defined

$$\Gamma(A) = \inf\{\gamma(B) : B \text{ has the same Turing degree as } A\}.$$ 

A smaller $\Gamma$ value means that $A$ is further away from computable.

**Example**

An oracle $A$ is called computably dominated if every function that $A$ computes is below a computable function. They show:

- If $A$ is random and computably dominated, then $\Gamma(A) = 1/2$.
- If $A$ is not computably dominated then $\Gamma(A) = 0$. 
Fact (Hirschfeldt et al., 2013)

If $\Gamma(A) > 1/2$ then $A$ is computable (so that $\Gamma(A) = 1$).

Idea:

- Obtain $B$ of the same Turing degree as $A$ by “padding”:
  - “Stretch” the value $A(n)$ over the whole interval $I_n = [(n - 1)!, n!]$.
- Since $\gamma(B) > 1/2$ there is a computable $R$ agreeing with $B$ on more than half of the bits in almost every interval $I_n$.
- So for almost all $n$, the bit $A(n)$ equals the majority of values $R(k)$ where $k \in I_n$. 
The $\Gamma$-question

**Question ($\Gamma$-question, Andrews et al., 2013)**

Is there a set $A \subseteq \mathbb{N}$ such that $0 < \Gamma(A) < 1/2$?
New examples towards answering the question

Recall: Γ-question, Andrews et al., 2013

Is there a set $A \subseteq \mathbb{N}$ such that $0 < \Gamma(A) < 1/2$?

Summary of previously known examples:

<table>
<thead>
<tr>
<th>$\Gamma(A)$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma(A) = 0$</td>
<td>$A$ non computably dominated or $A$ PA</td>
</tr>
<tr>
<td>$\Gamma(A) = 1/2$</td>
<td>$A$ low for Schnorr; $A$ random &amp; comp. dominated</td>
</tr>
<tr>
<td>$\Gamma(A) = 1$</td>
<td>$A$ computable</td>
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</table>

- Towards answering the question, we obtain natural classes of oracles with $\Gamma$ value $1/2$, and with $\Gamma$ value $0$.
- This yields new examples for both cases.
Weakly Schnorr engulfing

- We view oracles as infinite bit sequences, that is, elements of Cantor space $2^\mathbb{N}$.
- A $\Sigma^0_1$ set has the form $\bigcup_i [\sigma_i]$ for an effective sequence $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ of strings. $[\sigma]$ denotes the sequences extending $\sigma$.
- A Schnorr test is an effective sequence $(S_m)_{m \in \mathbb{N}}$ of $\Sigma^0_1$ sets in $2^\mathbb{N}$ such that
  - each $\lambda S_m$ is a computable real uniformly in $m$
  - $\lambda S_m \leq 2^{-m}$. ($\lambda$ is the usual uniform measure on $2^\mathbb{N}$.)
- Fact: $\bigcap_m S_m$ fails to contain all computable sets.

We can relativize these notions to an oracle $A$.

We say that $A$ is weakly Schnorr engulfing if $A$ computes a Schnorr test containing all the computable sets.

This highness property of oracles was introduced by Rupprecht (2010), in analogy with 1980s work in set theory (cardinal characteristics).
Examples of \( A \) such that \( \Gamma(A) \geq 1/2 \)

- The two known properties of \( A \) implying \( \Gamma(A) \geq 1/2 \) were:
  1. Computably dominated random, and
  2. low for Schnorr test:
     every \( A \)-Schnorr test is covered by a plain Schnorr test.

- Both properties imply non-weakly Schnorr engulfing.

- There is a non-weakly Schnorr engulfing set without any of these properties. (Kjos-Hanssen, Stephan and Terwijn, 2015).

So the following result yields new examples, answering Question 5.1 in Andrews et al.

**Theorem**

*Let \( A \) be not weakly Schnorr engulfing. Then \( \Gamma(A) \geq 1/2 \).*

*Proof:* Given \( B \leq_T A \) and rational \( \epsilon > 0 \), build an \( A \)-Schnorr test so that any set \( R \) passing it approximates \( B \) with asymptotic correctness \( \geq 1/2 - \epsilon \). This uses Chernoff bounds.
Characterization of w.S.e. via traces

An obvious question is whether conversely, $\Gamma(A) \geq 1/2$ implies that $A$ is not weakly Schnorr engulfing. We characterised w.S.e. towards obtaining an answer. Again this is analogous to earlier work in cardinal characteristics.

Let $H : \mathbb{N} \mapsto \mathbb{N}$ be computable with $\sum 1/H(n)$ finite. 

$\{T_n\}_{n \in \omega}$ is a small computable $H$-trace if

- $T_n$ is a uniformly computable finite set
- $\sum_n |T_n|/H(n)$ is finite and computable.

**Theorem**

$A$ is weakly Schnorr engulfing iff for some computable function $H$, there is an $A$-computable small $H$-trace capturing every computable function bounded by $H$. 
Version of $\Gamma$ in computational complexity

Fix an alphabet $\Sigma$. For $Z, A \subseteq \Sigma^*$ let

$$\rho(Z) = \liminf_n \frac{|Z \cap \Sigma^{\leq n}|}{|\Sigma^{\leq n}|}$$

$$\gamma_{\text{poly}}(A) = \sup_{R \text{ poly time computable}} \rho(\{w : A(w) = R(w)\})$$

$$\Gamma_{\text{poly}}(A) = \inf \{ \gamma_{\text{poly}}(B) : B \equiv_T^p A \}.$$ 

- The basic facts from computability used above need to be re-examined in the context of complexity theory.
- We only know at present that the values $\Gamma_{\text{poly}}(A)$ can be each of $0, \frac{1}{|\Sigma|}, 1$. 
Examples of $\Gamma(A) = 0$: infinitely often equal

We know that $A \subseteq \mathbb{N}$ not computably dominated implies $\Gamma(A) = 0$.

- We say $g : \mathbb{N} \to \mathbb{N}$ is infinitely often equal (i.o.e.) if $\exists \infty n \ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.
- We say that $A \subseteq \mathbb{N}$ is i.o.e. if $A$ computes function $g$ that is i.o.e.

*Surprising fact:* $A$ is i.o.e. $\iff A$ not computably dominated.

$\Rightarrow$ Suppose $A$ computes a function $g$ that equals infinitely often to every computable function. Then no computable function bounds $g$.

$\Leftarrow$ Idea. Suppose $A$ computes a function $g$ that is dominated by no computable function. Then $g$ is infinitely often above the halting time of any computable total function.
New Examples of $\Gamma(A) = 0$: weaken infinitely often equal

We know $A$ not computably dominated implies $\Gamma(A) = 0$.

Recall

We say that $A$ is infinitely often equal (i.o.e.) if $A$ computes a function $g$ such that $\exists^\infty n \ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.

We can weaken this:

Let $H : \mathbb{N} \to \mathbb{N}$ be computable. We say that $A$ is $H$-infinitely often equal if $A$ computes a function $g$ such that $\exists^\infty n \ f(n) = g(n)$ for each computable function $f$ bounded by $H$.

This appears to get harder for $A$ the faster $H$ grows.
New example of $\Gamma(A) = 0$

Let $H : \mathbb{N} \to \mathbb{N}$ be computable. We say that $A \subseteq \mathbb{N}$ is $H$-infinitely often equal if $A$ computes a function $g$ such that $\exists^\infty n f(n) = g(n)$ for each computable function $f$ bounded by $H$.

**Theorem**

Let $A$ be $2^{(\alpha^n)}$-i.o.e. for some $\alpha > 1$. Then $\Gamma(A) = 0$.

Previously known examples of sets $A$ with $\Gamma(A) = 0$:

- not computably dominated, and
- degree of a completion of Peano arithmetic (PA for short).

If $A$ is in one of these classes, for any computable bound $H$, $A$ can compute an $H$-i.o.e. function.

Given a computable $H \geq 2$, we can build an $H$-i.o.e. set $A$ that is computably dominated, and not PA. So we have a new example of $\Gamma(A) = 0$ (using Rupprecht (2010)).
New example of $\Gamma(A) = 0$

(Recall: $A$ is $H$-infinitely often equal if $A$ computes a function $g$ such that $\exists \infty nf(n) = g(n)$ for each computable function $f$ bounded by $H$.)

**Theorem**

Let $A$ be $2^{(\alpha^n)}$-i.o.e. for some computable $\alpha > 1$. Then $\Gamma(A) = 0$.

Proof sketch. First step: Let $f$ be $2^{(\alpha^n)}$-i.o.e. Then for any $k \in \mathbb{N}$, $f$ computes a function $g$ that is $2^{(k^n)}$-i.o.e.

$$f(0) \ f(1) \ f(2) \ f(3) \ f(4) \ f(5) \ldots \text{ i.o.e. every comp. funct. } \leq 2^{(\alpha^n)}$$

$$\rightarrow f(0)f(2)f(4) \ldots \text{ i.o.e. every comp. funct. } \leq n \mapsto 2^{(\alpha^{2n})}$$

or

$$f(1)f(3)f(5) \ldots \text{ i.o.e. every comp. funct. } \leq n \mapsto 2^{(\alpha^{2n+1})}$$

Iterating this $\rightarrow f \geq_T g$ which i.o.e. every comp. funct. $\leq 2^{(k^n)}$
Proof sketch. Second step: \( g \) is \( 2^{(k^n)} \)-i.o.e. implies \( g \geq_T Z \) with \( \Gamma(Z) \leq 1/k \).

\[
\begin{align*}
g(0) & \quad g(1) & \quad \ldots & \quad g(n) & \quad \ldots \\
= & \quad = & \quad \ldots & \quad = & \quad \ldots \\
Z : \quad \underbrace{\sigma_0} & \quad \underbrace{\sigma_1} & \quad \ldots & \quad \underbrace{\sigma_n} & \quad \ldots \\
|\sigma_0| = k^0 & \quad |\sigma_1| = k^1 & \quad & \quad |\sigma_n| = k^n
\end{align*}
\]

Computable \( R \):

\[
\begin{align*}
\tau_0 & \quad \tau_1 & \quad \ldots & \quad \tau_n & \quad \ldots \\
\downarrow & \quad (\text{bit flip}) & \quad & \quad & \quad \\
\overline{R} : \quad \overline{\tau_0} & \quad \overline{\tau_1} & \quad \ldots & \quad \overline{\tau_n} & \quad \ldots \\
= & \quad = & \quad = & \quad = \\
\end{align*}
\]

\( j \) equals \( g \) infinitely often. Then for infinitely many \( n \), \( \tau_n(i) \neq \sigma_n(i) \) everywhere. We have

\[
|\tau_n| \geq (k - 1) \sum_{i < n} |\tau_i|
\]

Then the \( \lim \inf \) of fraction of places where \( R \) agrees with \( Z \) is bounded by \( 1/k \).
Infinitely often equal: hierarchy

It is interesting to study infinite often equality for its own sake.

**Question**

Let $H$ be a computable bound. Can we always find $H' \gg H$ such that some $f$ is $H$-i.o.e. but $f$ computes no function that is $H'$-i.o.e.? 

First step: What about $H$-i.o.e. for $H$ constant?

- $X$ computable $\rightarrow$ $X$ not 2-i.o.e. $\rightarrow$ $X$ not $c$-i.o.e. for $c \in \mathbb{N}$
- $X$ not 2-i.o.e. $\rightarrow$ $X$ computable.
- $X$ not 3-i.o.e. $\rightarrow$ ?

\[
\begin{align*}
Z \in 2^\mathbb{N}: & \quad 0010101000100100101 \\
R \text{ computable}: & \quad 1101010111011011010 \\
\hline
Z \in 3^\mathbb{N}: & \quad 0210122002100102122 \\
R \text{ computable}: & \quad 1102010111011211210
\end{align*}
\]
Infinitely often equal: constant bound

For any \( c \in \mathbb{N} \), we can show \( X \) not \( c \text{-i.o.e.} \) \( \rightarrow X \) computable. Let \( c = 3 \).

For \( Z \in 2^\omega \), let \( \#_2^Z : \omega^2 \rightarrow \omega \) the function which on \( a, b \in \mathbb{N} \) returns \( |Z \cap \{a, b\}| \). Note that \( \#_2^Z \) can take three different values: \( 0, 1 \) and \( 2 \).

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**Theorem (Kummer)**

Suppose \( Z \) is an oracle such that \( \#_3^Z \) is traceable via some trace \( \{T_n\}_{n \in \omega} \), where each \( T_n \) is c.e. uniformly in \( n \) and \( |T_n| \leq 3 \). Then \( Z \) is computable.

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Example:

\[
\begin{align*}
Z &= 0 1 0 0 1 1 0 1 \cdots \\
\#_3^Z(2, 3) &\in \{0, 2\} \\
\#_3^Z(1, 4) &\in \{1, 2\} \\
\#_3^Z(3, 7) &\in \{0, 1\} \\
\end{align*}
\]
Infinitely often equal: implications

Known implications:

\[ c \text{-i.o.e. for } c \geq 2 \leftarrow H(n)\text{-i.o.e with } H \text{ computable order function s.t. } \sum_n \frac{1}{H(n)} = \infty \]

\[ \uparrow \uparrow \]

not computable \[ H(n)\text{-i.o.e with } H \text{ computable order function s.t. } \sum_n \frac{1}{H(n)} < \infty \]

We don’t know that there is a proper hierarchy for functions \( H \) with \( \infty > \sum_n 1/H(n) \).
References

- These slides on Nies’ web page.