A unifying approach to the Gamma question

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Lowness paradigms

Given a set $A \subseteq \mathbb{N}$. How close is A to being computable?

Several paradigms have been suggested and studied.

- ▶ A has little power as a Turing oracle.
- Many oracles compute A.

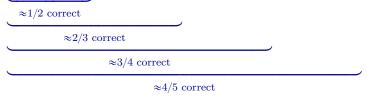
A recent paradigm: A is coarsely computable. This means there is a computable set R such that the asymptotic density of $\{n: A(n) = R(n)\}$ equals 1.

Reference: Downey, Jockusch, and Schupp, Asymptotic density and computably enumerable sets, Journal of Mathematical Logic, 13, No. 2 (2013)

The γ -value of a set $A \subseteq \mathbb{N}$

A computable set R tries to approximate a complicated set A:

- $A: 100100100100\,000101001001\,010101111010\,101010101111$

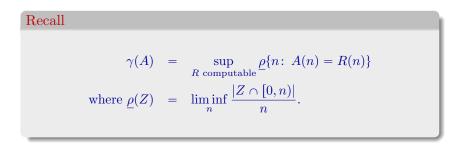


Take sup of the asymptotic correctness over all computable R's:

$$\gamma(A) = \sup_{\substack{R \text{ computable}}} \underline{\rho}\{n \colon A(n) = R(n)\}$$

where $\underline{\rho}(Z) = \liminf_{n} \frac{|Z \cap [0, n)|}{n}.$

Some examples of values $\gamma(A)$



Some possible values

 $A \text{ computable } \Rightarrow \gamma(A) = 1$ $A \text{ random } \Rightarrow \gamma(A) = 1/2.$

$\Gamma\text{-value}$ of a Turing degree

Andrews, Cai, Diamondstone, Jockusch and Lempp (2013) looked at Turing degrees, rather than sets. They defined

 $\Gamma(A) = \inf\{\gamma(B): B \text{ has the same Turing degree as } A\}.$

A smaller Γ value means that A is further away from computable.

Example

An oracle A is called computably dominated if every function that A computes is below a computable function. *They show:*

- If A is random and computably dominated, then $\Gamma(A) = 1/2$.
- If A is not computably dominated then $\Gamma(A) = 0$.

$\Gamma(A) > 1/2$ implies $\Gamma(A) = 1$

Fact (Hirschfeldt et al., 2013)

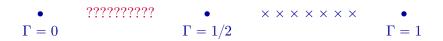
If $\Gamma(A) > 1/2$ then A is computable (so that $\Gamma(A) = 1$).

Idea:

- Obtain B of the same Turing degree as A by "padding":
- "Stretch" the value A(n) over the whole interval $I_n = [(n-1)!, n!)$.
- Since $\gamma(B) > 1/2$ there is a computable R agreeing with B on more than half of the bits in almost every interval I_n .
- ▶ So for almost all n, the bit A(n) equals the majority of values R(k) where $k \in I_n$.

The Γ -question

Question (Γ -question, Andrews et al., 2013) Is there a set $A \subseteq \mathbb{N}$ such that $0 < \Gamma(A) < 1/2$?



New examples towards answering the question

Recall: Γ -question, Andrews et al., 2013 Is there a set $A \subseteq \mathbb{N}$ such that $0 < \Gamma(A) < 1/2$?

Summary of previously known examples:

$\Gamma(A) = 0$	A non computably dominated or A PA
$\Gamma(A) = 1/2$	A low for Schnorr; A random & comp. dominated
$\Gamma(A) = 1$	A computable

- Towards answering the question, we obtain natural classes of oracles with Γ value 1/2, and with Γ value 0.
- This yields new examples for both cases.

Weakly Schnorr engulfing

- ► We view oracles as infinite bit sequences, that is, elements of Cantor space 2^N.
- A Σ_1^0 set has the form $\bigcup_i [\sigma_i]$ for an effective sequence $\langle \sigma_i \rangle_{i \in \mathbb{N}}$ of strings. $[\sigma]$ denotes the sequences extending σ .
- ► A Schnorr test is an effective sequence $(S_m)_{m\in\mathbb{N}}$ of Σ_1^0 sets in $2^{\mathbb{N}}$ such that
 - each λS_m is a computable real uniformly in m
 - $-\lambda S_m \leq 2^{-m}$. (λ is the usual uniform measure on $2^{\mathbb{N}}$.)
- Fact: $\bigcap_m S_m$ fails to contain all computable sets.

We can relativize these notions to an oracle A.

We say that A is weakly Schnorr engulfing if A computes a Schnorr test containing all the computable sets.

This highness property of oracles was introduced by Rupprecht (2010), in analogy with 1980s work in set theory (cardinal characteristics).

Examples of A such that $\Gamma(A) \ge 1/2$

- The two known properties of A implying $\Gamma(A) \ge 1/2$ were:
 - (1) Computably dominated random, and
 - (2) low for Schnorr test:

every A-Schnorr test is covered by a plain Schnorr test.

- ▶ Both properties imply non-weakly Schnorr engulfing.
- There is a non-weakly Schnorr engulfing set without any of these properties. (Kjos-Hanssen, Stephan and Terwijn, 2015).

So the following result yields new examples, answering Question 5.1 in Andrews et al.

Theorem

Let A be not weakly Schnorr engulfing. Then $\Gamma(A) \ge 1/2$.

Proof: Given $B \leq_{\mathrm{T}} A$ and rational $\epsilon > 0$, build an A-Schnorr test so that any set R passing it approximates B with asymptotic correctness $\geq 1/2 - \epsilon$. This uses Chernoff bounds.

Characterization of w.S.e. via traces

An obvious question is whether conversely, $\Gamma(A) \ge 1/2$ implies that A is not weakly Schnorr engulfing. We characterised w.S.e. towards obtaining an answer. Again this is analogous to earlier work in cardinal characteristics.

Let $H : \mathbb{N} \mapsto \mathbb{N}$ be computable with $\sum 1/H(n)$ finite. $\{T_n\}_{n \in \omega}$ is a small computable *H*-trace if

- T_n is a uniformly computable finite set
- $\sum_{n} |T_n|/H(n)$ is finite and computable.

Theorem

A is weakly Schnorr engulfing iff for some computable function H, there is an A-computable small H-trace capturing every computable function bounded by H.

Version of Γ in computational complexity

Fix an alphabet Σ . For $Z, A \subseteq \Sigma^*$ let

$$\underline{\rho}(Z) = \liminf_{n} \frac{|Z \cap \Sigma^{\leq n}|}{|\Sigma^{\leq n}|}$$

$$\gamma_{poly}(A) = \sup_{\substack{R \text{ poly time computable} \\ P_{poly}(A)}} \underline{\rho}(\{w \colon A(w) = R(w)\})$$

$$\Gamma_{poly}(A) = \inf\{\gamma_{poly}(B) \colon B \equiv_{T}^{p} A\}.$$

- ➤ The basic facts from computability used above need to be re-examined in the context of complexity theory.
- We only know at present that the values $\Gamma_{poly}(A)$ can be each of $0, \frac{1}{|\Sigma|}, 1$.

Examples of $\Gamma(A) = 0$: infinitely often equal

We know that $A \subseteq \mathbb{N}$ not computably dominated implies $\Gamma(A) = 0$.

- ▶ We say $g : \mathbb{N} \to \mathbb{N}$ is infinitely often equal (i.o.e.) if $\exists^{\infty}n \ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.
- ▶ We say that $A \subseteq \mathbb{N}$ is i.o.e. if A computes function g that is i.o.e.

Surprising fact: A is i.o. $e \Leftrightarrow A$ not computably dominated.

 \Rightarrow Suppose A computes a function g that equals infinitely often to every computable function. Then no computable function bounds g.

 \leftarrow *Idea*. Suppose A computes a function g that is dominated by no computable function. Then g is infinitely often above the halting time of any computable total function.

New Examples of $\Gamma(A) = 0$: weaken infinitely often equal

We know A not computably dominated implies $\Gamma(A) = 0$.

Recall

We say that A is infinitely often equal (i.o.e.) if A computes a function g such that $\exists^{\infty} n \ f(n) = g(n)$ for each computable function $f : \mathbb{N} \to \mathbb{N}$.

We can weaken this:

Let $H: \mathbb{N} \to \mathbb{N}$ be computable. We say that A is H-infinitely often equal if A computes a function g such that $\exists^{\infty} n \ f(n) = g(n)$ for each computable function f bounded by H.

This appears to get harder for A the faster H grows.

New example of $\Gamma(A) = 0$

Let $H : \mathbb{N} \to \mathbb{N}$ be computable. We say that $A \subseteq \mathbb{N}$ is H-infinitely often equal if A computes a function g such that $\exists^{\infty} n f(n) = g(n)$ for each computable function f bounded by H.

Theorem

Let A be $2^{(\alpha^n)}$ -i.o.e. for some $\alpha > 1$.

Then $\Gamma(A) = 0$.

Previously known examples of sets A with $\Gamma(A) = 0$:

- ▶ not computably dominated, and
- degree of a completion of Peano arithmetic (PA for short).

If A is in one of these classes, for any computable bound H, A can compute an H-i.o.e. function.

Given a computable $H \ge 2$, we can build an *H*-i.o.e. set *A* that is computably dominated, and not PA. So we have a new example of $\Gamma(A) = 0$ (using Rupprecht (2010)).

New example of $\Gamma(A) = 0$

(Recall: A is *H*-infinitely often equal if A computes a function g such that $\exists^{\infty} nf(n) = g(n)$ for each computable function f bounded by H.)

Theorem

Let A be $2^{(\alpha^n)}$ -i.o.e. for some computable $\alpha > 1$.

Then $\Gamma(A) = 0$.

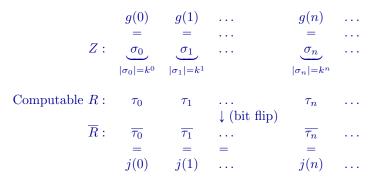
Proof sketch. First step: Let f be $2^{(\alpha^n)}$ -i.o.e. Then for any $k \in \mathbb{N}$, f computes a function g that is $2^{(k^n)}$ -i.o.e.

f(0) f(1) f(2) f(3) f(4) f(5) ... i.o.e. every comp. funct. $\leq 2^{(\alpha^n)}$

 $\rightarrow \qquad f(0)f(2)f(4)\dots \quad \text{i.o.e. every comp. funct.} \quad \leqslant n \mapsto 2^{(\alpha^{2n})} \\ \text{or} \quad f(1)f(3)f(5)\dots \quad \text{i.o.e. every comp. funct.} \quad \leqslant n \mapsto 2^{(\alpha^{2n+1})}$

Iterating this $\rightarrow f \ge_T g$ which i.o.e. every comp. funct. $\leq 2^{(k^n)}$

Proof sketch. Second step: g is $2^{(k^n)}$ -i.o.e. implies $g \ge_T Z$ with $\Gamma(Z) \le 1/k$.



j equals g infinitely often. Then for infinitely many n, $\tau_n(i) \neq \sigma_n(i)$ everywhere. We have

$$|\tau_n| \ge (k-1)\sum_{i < n} |\tau_i|$$

Then the lim inf of fraction of places where R agrees with Z is bounded by 1/k.

Infinitely often equal: hierarchy

It is interesting to study infinite often equality for its own sake.

Question

Let H be a computable bound. Can we always find H' >> Hsuch that some f is H-i.o.e. but f computes no function that is H'-i.o.e. ?

First step : What about *H*-i.o.e. for *H* constant? *X* computable \rightarrow *X* not 2-i.o.e. \rightarrow *X* not *c*-i.o.e. for $c \in \mathbb{N}$ *X* not 2-i.o.e. \rightarrow *X* computable. *X* not 3-i.o.e. \rightarrow ?

$Z\in 2^{\mathbb{N}}$:	0010101000100100101
R computable :	1101010111011011010
$Z \in 3^{\mathbb{N}}$:	0210122002100102122
R computable :	1102010111011211210

Infinitely often equal: constant bound

For any $c \in \mathbb{N}$, we can show X not c-i.o.e. $\to X$ computable. Let c = 3. For $Z \in 2^{\omega}$, let $\#_2^Z : \omega^2 \to \omega$ the function which on $a, b \in \mathbb{N}$ returns $|Z \cap \{a, b\}|$. Note that $\#_2^Z$ can take three different values : 0, 1 and 2.

Theorem (Kummer)

Suppose Z is an oracle such that $\#_3^Z$ is traceable via some trace $\{T_n\}_{n\in\omega}$, where each T_n is c.e. uniformly in n and $|T_n| \leq 3$. Then Z is computable.

Example:

$$\begin{array}{rcl} \#^Z_3(2,3) & \in & \{0,2\} \\ \#^Z_3(1,4) & \in & \{1,2\} \\ \#^Z_3(3,7) & \in & \{0,1\} \end{array}$$

Infinitely often equal: implications

Known implications:

c-i.o.e. for $c \ge 2 \leftarrow H(n)$ -i.o.e with H computable order function s.t. $\sum_n \frac{1}{H(n)} = \infty$ \uparrow not computable H(n)-i.o.e with H computable order function s.t. $\sum_n \frac{1}{H(n)} < \infty$

We don't know that there is a proper hierarchy for functions H with $\infty > \sum_n 1/H(n)$.

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 Effective correspondent to Cardinal characteristics in Cichoń's diagram. PhD Thesis, Univ of Michigan, 2010.
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- ▶ These slides on Nies' web page.