## A unifying approach to the Gamma question

Benoit Monin

André Nies





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## Lowness paradigms

Given a set  $A \subseteq \mathbb{N}$ . How close is A to being computable?

Several paradigms have been suggested and studied.

- ▶ A has little power as a Turing oracle.
- Many oracles compute A.

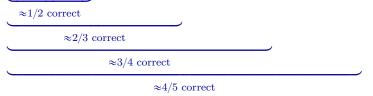
A recent paradigm: A is coarsely computable. This means there is a computable set R such that the asymptotic density of  $\{n: A(n) = R(n)\}$  equals 1.

Reference: Downey, Jockusch, and Schupp, Asymptotic density and computably enumerable sets, Journal of Mathematical Logic, 13, No. 2 (2013)

### The $\gamma$ -value of a set $A \subseteq \mathbb{N}$

A computable set R tries to approximate a complicated set A:

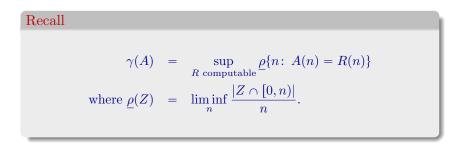
- $A: 100100100100\,000101001001\,010101111010\,101010101111$



Take sup of the asymptotic correctness over all computable R's:

$$\gamma(A) = \sup_{\substack{R \text{ computable}}} \underline{\rho}\{n \colon A(n) = R(n)\}$$
  
where  $\underline{\rho}(Z) = \liminf_{n} \frac{|Z \cap [0, n)|}{n}.$ 

# Some examples of values $\gamma(A)$



#### Some possible values

 $A \text{ computable } \Rightarrow \gamma(A) = 1$  $A \text{ random } \Rightarrow \gamma(A) = 1/2.$ 

# $\Gamma\text{-value}$ of a Turing degree

Andrews, Cai, Diamondstone, Jockusch and Lempp (2013) looked at Turing degrees, rather than sets. They defined

 $\Gamma(A) = \inf\{\gamma(B): B \text{ has the same Turing degree as } A\}.$ 

A smaller  $\Gamma$  value means that A is further away from computable.

#### Example

An oracle A is called computably dominated if every function that A computes is below a computable function. *They show:* 

- If A is random and computably dominated, then  $\Gamma(A) = 1/2$ .
- If A is not computably dominated then  $\Gamma(A) = 0$ .

# $\Gamma(A) > 1/2$ implies $\Gamma(A) = 1$

Fact (Hirschfeldt et al., 2013)

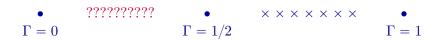
If  $\Gamma(A) > 1/2$  then A is computable (so that  $\Gamma(A) = 1$ ).

### Idea:

- Obtain B of the same Turing degree as A by "padding":
- "Stretch" the value A(n) over the whole interval  $I_n = [(n-1)!, n!)$ .
- Since  $\gamma(B) > 1/2$  there is a computable R agreeing with B on more than half of the bits in almost every interval  $I_n$ .
- ▶ So for almost all n, the bit A(n) equals the majority of values R(k) where  $k \in I_n$ .

## The $\Gamma$ -question

Question ( $\Gamma$ -question, Andrews et al., 2013) Is there a set  $A \subseteq \mathbb{N}$  such that  $0 < \Gamma(A) < 1/2$ ?



New examples towards answering the question

Recall:  $\Gamma$ -question, Andrews et al., 2013 Is there a set  $A \subseteq \mathbb{N}$  such that  $0 < \Gamma(A) < 1/2$ ?

Summary of previously known examples:

| $\Gamma(A) = 0$   | A non computably dominated or $A$ PA            |
|-------------------|---|
| $\Gamma(A) = 1/2$ | A low for Schnorr; $A$ random & comp. dominated |
| $\Gamma(A) = 1$   | A computable                                    |

- Towards answering the question, we obtain natural classes of oracles with  $\Gamma$  value 1/2, and with  $\Gamma$  value 0.
- This yields new examples for both cases.

# Weakly Schnorr engulfing

- ► We view oracles as infinite bit sequences, that is, elements of Cantor space 2<sup>N</sup>.
- A  $\Sigma_1^0$  set has the form  $\bigcup_i [\sigma_i]$  for an effective sequence  $\langle \sigma_i \rangle_{i \in \mathbb{N}}$  of strings.  $[\sigma]$  denotes the sequences extending  $\sigma$ .
- ► A Schnorr test is an effective sequence  $(S_m)_{m\in\mathbb{N}}$  of  $\Sigma_1^0$  sets in  $2^{\mathbb{N}}$  such that
  - each  $\lambda S_m$  is a computable real uniformly in m
  - $-\lambda S_m \leq 2^{-m}$ . ( $\lambda$  is the usual uniform measure on  $2^{\mathbb{N}}$ .)
- Fact:  $\bigcap_m S_m$  fails to contain all computable sets.

We can relativize these notions to an oracle A.

We say that A is weakly Schnorr engulfing if A computes a Schnorr test containing all the computable sets.

This highness property of oracles was introduced by Rupprecht (2010), in analogy with 1980s work in set theory (cardinal characteristics).

Examples of A such that  $\Gamma(A) \ge 1/2$ 

- The two known properties of A implying  $\Gamma(A) \ge 1/2$  were:
  - (1) Computably dominated random, and
  - (2) low for Schnorr test:

every A-Schnorr test is covered by a plain Schnorr test.

- ▶ Both properties imply non-weakly Schnorr engulfing.
- There is a non-weakly Schnorr engulfing set without any of these properties. (Kjos-Hanssen, Stephan and Terwijn, 2015).

So the following result yields new examples, answering Question 5.1 in Andrews et al.

### Theorem

Let A be not weakly Schnorr engulfing. Then  $\Gamma(A) \ge 1/2$ .

*Proof:* Given  $B \leq_{\mathrm{T}} A$  and rational  $\epsilon > 0$ , build an A-Schnorr test so that any set R passing it approximates B with asymptotic correctness  $\geq 1/2 - \epsilon$ . This uses Chernoff bounds.

## Characterization of w.S.e. via traces

An obvious question is whether conversely,  $\Gamma(A) \ge 1/2$  implies that A is not weakly Schnorr engulfing. We characterised w.S.e. towards obtaining an answer. Again this is analogous to earlier work in cardinal characteristics.

Let  $H : \mathbb{N} \mapsto \mathbb{N}$  be computable with  $\sum 1/H(n)$  finite.  $\{T_n\}_{n \in \omega}$  is a small computable *H*-trace if

- $T_n$  is a uniformly computable finite set
- $\sum_{n} |T_n|/H(n)$  is finite and computable.

#### Theorem

A is weakly Schnorr engulfing iff for some computable function H, there is an A-computable small H-trace capturing every computable function bounded by H.

## Version of $\Gamma$ in computational complexity

Fix an alphabet  $\Sigma$ . For  $Z, A \subseteq \Sigma^*$  let

$$\underline{\rho}(Z) = \liminf_{n} \frac{|Z \cap \Sigma^{\leq n}|}{|\Sigma^{\leq n}|}$$

$$\gamma_{poly}(A) = \sup_{\substack{R \text{ poly time computable} \\ P_{poly}(A)}} \underline{\rho}(\{w \colon A(w) = R(w)\})$$

$$\Gamma_{poly}(A) = \inf\{\gamma_{poly}(B) \colon B \equiv_{T}^{p} A\}.$$

- ➤ The basic facts from computability used above need to be re-examined in the context of complexity theory.
- We only know at present that the values  $\Gamma_{poly}(A)$  can be each of  $0, \frac{1}{|\Sigma|}, 1$ .

# Examples of $\Gamma(A) = 0$ : infinitely often equal

We know that  $A \subseteq \mathbb{N}$  not computably dominated implies  $\Gamma(A) = 0$ .

- ▶ We say  $g : \mathbb{N} \to \mathbb{N}$  is infinitely often equal (i.o.e.) if  $\exists^{\infty}n \ f(n) = g(n)$  for each computable function  $f : \mathbb{N} \to \mathbb{N}$ .
- ▶ We say that  $A \subseteq \mathbb{N}$  is i.o.e. if A computes function g that is i.o.e.

Surprising fact: A is i.o.  $e \Leftrightarrow A$  not computably dominated.

 $\Rightarrow$  Suppose A computes a function g that equals infinitely often to every computable function. Then no computable function bounds g.

 $\leftarrow$  *Idea*. Suppose A computes a function g that is dominated by no computable function. Then g is infinitely often above the halting time of any computable total function.

New Examples of  $\Gamma(A) = 0$ : weaken infinitely often equal

We know A not computably dominated implies  $\Gamma(A) = 0$ .

#### Recall

We say that A is infinitely often equal (i.o.e.) if A computes a function g such that  $\exists^{\infty} n \ f(n) = g(n)$  for each computable function  $f : \mathbb{N} \to \mathbb{N}$ .

We can weaken this:

Let  $H: \mathbb{N} \to \mathbb{N}$  be computable. We say that A is H-infinitely often equal if A computes a function g such that  $\exists^{\infty} n \ f(n) = g(n)$ for each computable function f bounded by H.

This appears to get harder for A the faster H grows.

# New example of $\Gamma(A) = 0$

Let  $H : \mathbb{N} \to \mathbb{N}$  be computable. We say that  $A \subseteq \mathbb{N}$  is H-infinitely often equal if A computes a function g such that  $\exists^{\infty} n f(n) = g(n)$ for each computable function f bounded by H.

#### Theorem

Let A be  $2^{(\alpha^n)}$ -i.o.e. for some  $\alpha > 1$ .

Then  $\Gamma(A) = 0$ .

Previously known examples of sets A with  $\Gamma(A) = 0$ :

- ▶ not computably dominated, and
- degree of a completion of Peano arithmetic (PA for short).

If A is in one of these classes, for any computable bound H, A can compute an H-i.o.e. function.

Given a computable  $H \ge 2$ , we can build an *H*-i.o.e. set *A* that is computably dominated, and not PA. So we have a new example of  $\Gamma(A) = 0$  (using Rupprecht (2010)).

# New example of $\Gamma(A) = 0$

(Recall: A is *H*-infinitely often equal if A computes a function g such that  $\exists^{\infty} nf(n) = g(n)$  for each computable function f bounded by H.)

#### Theorem

Let A be  $2^{(\alpha^n)}$ -i.o.e. for some computable  $\alpha > 1$ .

Then  $\Gamma(A) = 0$ .

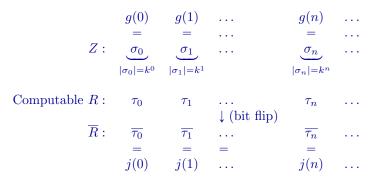
Proof sketch. First step: Let f be  $2^{(\alpha^n)}$ -i.o.e. Then for any  $k \in \mathbb{N}$ , f computes a function g that is  $2^{(k^n)}$ -i.o.e.

f(0) f(1) f(2) f(3) f(4) f(5) ... i.o.e. every comp. funct.  $\leq 2^{(\alpha^n)}$ 

 $\rightarrow \qquad f(0)f(2)f(4)\dots \quad \text{i.o.e. every comp. funct.} \quad \leqslant n \mapsto 2^{(\alpha^{2n})} \\ \text{or} \quad f(1)f(3)f(5)\dots \quad \text{i.o.e. every comp. funct.} \quad \leqslant n \mapsto 2^{(\alpha^{2n+1})}$ 

Iterating this  $\rightarrow f \ge_T g$  which i.o.e. every comp. funct.  $\leq 2^{(k^n)}$ 

Proof sketch. Second step: g is  $2^{(k^n)}$ -i.o.e. implies  $g \ge_T Z$  with  $\Gamma(Z) \le 1/k$ .



j equals g infinitely often. Then for infinitely many n,  $\tau_n(i) \neq \sigma_n(i)$  everywhere. We have

$$|\tau_n| \ge (k-1)\sum_{i < n} |\tau_i|$$

Then the lim inf of fraction of places where R agrees with Z is bounded by 1/k.

# Infinitely often equal: hierarchy

It is interesting to study infinite often equality for its own sake.

### Question

Let H be a computable bound. Can we always find H' >> Hsuch that some f is H-i.o.e. but f computes no function that is H'-i.o.e. ?

First step : What about *H*-i.o.e. for *H* constant? *X* computable  $\rightarrow$  *X* not 2-i.o.e.  $\rightarrow$  *X* not *c*-i.o.e. for  $c \in \mathbb{N}$  *X* not 2-i.o.e.  $\rightarrow$  *X* computable. *X* not 3-i.o.e.  $\rightarrow$  ?

| $Z\in 2^{\mathbb{N}}$ :  | 0010101000100100101 |
|--------------------------|---------------------|
| R computable :           | 1101010111011011010 |
| $Z \in 3^{\mathbb{N}}$ : | 0210122002100102122 |
| R computable :           | 1102010111011211210 |

## Infinitely often equal: constant bound

For any  $c \in \mathbb{N}$ , we can show X not c-i.o.e.  $\to X$  computable. Let c = 3. For  $Z \in 2^{\omega}$ , let  $\#_2^Z : \omega^2 \to \omega$  the function which on  $a, b \in \mathbb{N}$ returns  $|Z \cap \{a, b\}|$ . Note that  $\#_2^Z$  can take three different values : 0, 1 and 2.

#### Theorem (Kummer)

Suppose Z is an oracle such that  $\#_3^Z$  is traceable via some trace  $\{T_n\}_{n\in\omega}$ , where each  $T_n$  is c.e. uniformly in n and  $|T_n| \leq 3$ . Then Z is computable.

#### Example:

$$\begin{array}{rcl} \#^Z_3(2,3) & \in & \{0,2\} \\ \#^Z_3(1,4) & \in & \{1,2\} \\ \#^Z_3(3,7) & \in & \{0,1\} \end{array}$$

Infinitely often equal: implications

Known implications:

c-i.o.e. for  $c \ge 2 \leftarrow H(n)$ -i.o.e with H computable order function s.t.  $\sum_n \frac{1}{H(n)} = \infty$  $\uparrow$ not computable H(n)-i.o.e with H computable order function s.t.  $\sum_n \frac{1}{H(n)} < \infty$ 

We don't know that there is a proper hierarchy for functions H with  $\infty > \sum_n 1/H(n)$ .

## References

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- ▶ These slides on Nies' web page.