

Multiple recurrence and algorithmic randomness

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CCR 2015, Universität Heidelberg

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Plan

- ▶ Algorithmic randomness connects to ergodic theory via an effective study of “almost-everywhere” statements, such as Birkhoff’s 1939 theorem:

Let (X, μ, T) be a measure preserving system, and let $f: X \rightarrow \mathbb{R}$ is measurable. For μ -almost every x , the limit as $N \rightarrow \infty$ of the averages of $f \circ T^i(x)$ over $0 \leq i < N$, exists.

- ▶ We address these connections for the multiple recurrence theorem due to Furstenberg (J. Analyse Math., 1977). So far we only do this in the rather special case of shifts on Cantor space.

Some important papers connecting algorithmic randomness with ergodic theory

- ▶ V'yugin, TCS, 1999.

Shows that ML-randomness suffices for the effective Birkhoff theorem. (Note that $T: \subseteq X \rightarrow X$ only needs to be defined μ -a.e.)

- ▶ Franklin and Towsner, Moscow Math. J, recent.

Sharpness of V'yugin's result.

- ▶ Gacs, Hoyrup, Rojas, 2009; Galatolo, Hoyrup, Rojas, 2011.

General theory of computable probability spaces and computable measure preserving systems; Kolmogorov-Sinai entropy, etc.

Multiple recurrence

Classical theory

A measurable operator T on a probability space (X, \mathcal{B}, μ) is **measure preserving** if $\mu T^{-1}(A) = \mu A$ for each $A \in \mathcal{B}$.

The following is Furstenberg's multiple recurrence theorem (1977); see Furstenberg's book on recurrence, 2014 edition, Thm. 7.15.

Theorem

Let (X, \mathcal{B}, μ) be a probability space. Let T_1, \dots, T_k be commuting measure preserving operators on X . For each $P \in \mathcal{B}$ with $\mu P > 0$, there is $n > 0$ such that $\mu(\bigcap_i T_i^{-n}(P)) > 0$.

- ▶ In fact, he proves $0 < \liminf_N \frac{1}{N} \sum_{n=1}^N \mu(\bigcap_i T_i^{-n}(P))$.
- ▶ One can also strengthen to: **a.e.** $z \in P \exists n [z \in \bigcap_i T_i^{-n}(P)]$.

Kurtz \Rightarrow k -recurrence in clopen \mathcal{P}

In the following we work with $X = \{0, 1\}^{\mathbb{N}}$, and the shift operator $S : X \rightarrow X$ that takes the first bit off a sequence.

Definition

Let $\mathcal{P} \subseteq \{0, 1\}^{\mathbb{N}}$ be measurable, and $Z \in \{0, 1\}^{\mathbb{N}}$. We say that Z is **k -recurrent in \mathcal{P}** if $S^n(Z), S^{2n}(Z), \dots, S^{kn}(Z) \in \mathcal{P}$ for some $n \geq 1$, i.e.

$$Z \in \bigcap_{1 \leq i \leq k} S^{-ni}(P).$$

Proposition

Let $\mathcal{P} \subseteq \{0, 1\}^{\mathbb{N}}$ be clopen, $\mathcal{P} \neq \emptyset$.

Each Kurtz random Z is k -recurrent in \mathcal{P} , for each $k \geq 1$.

Proposition (again)

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be clopen, $\mathcal{P} \neq \emptyset$. Let Z be Kurtz random and $k \geq 1$.

There is $n \geq 1$ such that $Z \in \bigcap_{1 \leq i \leq k} S^{-ni}(\mathcal{P})$.

Suppose there is no such n . We define a null Π_1^0 class \mathcal{Q} containing Z .

- ▶ Let n_0 be least such that $\mathcal{P} = [F]^{<} for some set F of strings of length n_0 .$
- ▶ Let $n_t = n_0(k+1)^t$ for $t \geq 1$.
- ▶ Let $\mathcal{Q} = \{Y : \forall t [Y \notin \bigcap_{1 \leq i \leq k} S^{-int}(\mathcal{P})]\}$. Then $Z \in \mathcal{Q}$.

By definition of n_0 , the classes in the same intersection are independent, so we have for each t

$$\lambda(\{0,1\}^{\mathbb{N}} - \bigcap_{1 \leq i \leq k} S^{-int}(\mathcal{P})) = 1 - (\lambda\mathcal{P})^k < 1.$$

The Π_1^0 class \mathcal{Q} is the intersection of independent such classes ranging over all t . Therefore \mathcal{Q} is null.

Schnorr \Rightarrow k -recurrence in

Π_1^0 classes with positive computable measure

Theorem

Let $\mathcal{P} \subseteq \{0, 1\}^{\mathbb{N}}$ be a Π_1^0 class with $0 < p = \lambda\mathcal{P}$ a computable real. Each Schnorr random Z is k -recurrent in \mathcal{P} , for each $k \geq 1$.

This extends the previous argument. For each v we have an error set $G_v \subseteq \{0, 1\}^{\mathbb{N}}$. We make the sequence $\langle n_t \rangle$ grow much faster than before:

Let $n_0 = 1$. Let $n = n_t \geq (k + 1)n_{t-1}$ be so large that

$$\lambda(\mathcal{P}_n - \mathcal{P}) \leq 2^{-t-v-k}.$$

Define G_v so that $\langle G_v \rangle_{v \in \mathbb{N}}$ is a Schnorr test. If $Z \notin G_v$ for some v , we can apply the independence argument used for Kurtz

ML randomness \Rightarrow k -recurrence in Π_1^0 classes

Theorem

Let $\mathcal{P} \subseteq \{0,1\}^{\mathbb{N}}$ be a Π_1^0 class with $0 < \lambda\mathcal{P}$.

Each Martin-Löf random Z is k -recurrent in \mathcal{P} , for each $k \geq 1$.

Fix k . First we prove the assertion under the additional assumption that \mathcal{P} is large: $1 - 1/k < \lambda\mathcal{P}$.

For a string η and $u \leq |\eta|$, we write $S^u(\eta)$ for the string η with the first u bits removed.

Each Martin-Löf random Z is k -recurrent in \mathcal{P}

Let $B \subseteq 2^{<\omega}$ be a prefix-free c.e. set such that $[B]^\prec = \{0, 1\}^\mathbb{N} - \mathcal{P}$.

We define a uniformly c.e. sequence $\langle C^r \rangle$ of prefix free sets.

Let $C^0 = \{\emptyset\}$.

Suppose $r > 0$ and σ is enumerated into C^{r-1} at stage s (so $|\sigma| = s$).

Stage $t > (k + 1)s$: look for $\eta \succ \sigma$ a minimal string of length t such that $S^{si}(\eta) \in B_t$ for some $i \leq k$. Put η into C^r at stage t .

Let $q = k\lambda[B]^\prec$. Then $q < 1$ by hypothesis.

The local measure above σ of the η 's we put into C^r is at most q .

Inductively this implies:

For each $r \geq 0$ we have $\lambda[C^r]^\prec \leq q^r$.

If Z is not k -recurrent for \mathcal{P} then $Z \in \bigcap_r [C^r]^\prec$, so not ML-random.

General case

Theorem (again)

Let $\mathcal{P} \subseteq \{0, 1\}^{\mathbb{N}}$ be a Π_1^0 class with $0 < p = \lambda\mathcal{P}$.

Each Martin-Löf random Z is k -recurrent in \mathcal{P} , for each $k \geq 1$.

- ▶ If $1 - 1/k \geq \lambda\mathcal{P}$ (i.e., $1/k \leq \lambda[B]^{\complement}$ where $[B]^{\complement}$ is the complement of \mathcal{P}), then $\lambda[C^r]^{\complement}$ could easily be 1.
- ▶ To remedy this, we choose a finite set $D \subseteq B$ such that the set $\tilde{B} = B - D$ satisfies $1/k > \lambda[\tilde{B}]^{\complement}$.
- ▶ We modify the argument for the Kurtz case, using the complement of $[D]^{\complement}$ as the clopen set (called \mathcal{P} there).
- ▶ If Z passes a ML-test corresponding to the Kurtz test before, then the previous argument works with \tilde{B} .

Recurrence for k shift operators

- ▶ The space is $\mathcal{X} = \{0, 1\}^{\mathbb{N}^k}$
- ▶ For $1 \leq i \leq k$, the operator $T_i: \mathcal{X} \rightarrow \mathcal{X}$ takes a row of bits off in direction i .
- ▶ Z is **recurrent** in a class $\mathcal{P} \subseteq \mathcal{X}$ if $\exists n \forall i [Z \in T_i^{-n}(\mathcal{P})]$.

We can modify the methods above to show:

Theorem

Let $\mathcal{P} \subseteq \mathcal{X}$ be a Π_1^0 class with $0 < \lambda\mathcal{P}$. Let $Z \in \mathcal{X}$.

(a) Kurtz (b) Schnorr (c) ML-randomness of Z

implies that Z is recurrent in \mathcal{P} if

(a) \mathcal{P} is clopen (b) $\lambda\mathcal{P}$ is computable (c) for any \mathcal{P} .

Recurrence for compact systems

An example of a computable compact system is rotation of S^d of the unit circle C by $2\pi d$, for a computable irrational d . (This is ergodic and not weakly mixing.)

Question

Does multiple recurrence for powers of S_d hold for ML-random $z \in C$ and Π_1^0 classes of positive measure?

This would mean: for each Π_1^0 class $\mathcal{P} \subseteq C$ of positive measure, for each k ,

for a.e. z there is n such that $S_{din}(z) \in \mathcal{P}$ for $1 \leq i \leq k$.

Does it hold for computable compact systems in general?

General Conjecture

It is likely that an effective multiple recurrence theorem holds in full generality. If the system is not ergodic, as in the classic case, we have to require that $z \in \mathcal{P}$.

Conjecture

Let (X, μ) be a computable probability space. Let T_1, \dots, T_n be computable measure preserving transformation that commute pairwise. Let \mathcal{P} be a Π_1^0 class with $\mu\mathcal{P} > 0$.

If $z \in \mathcal{P}$ is ML-random then $\exists n \bigwedge_i T_i^n z \in \mathcal{P}$.

By the classic result this holds for weakly 2-random z . To get there for ML-random z , climb the Furstenberg-Zimmer tower? Jason Rute has pointed out that if $\mu\mathcal{P}$ is computable, then Schnorr randomness of z is sufficient.

A draft of this work is available on the 2015 Logic Blog.