An analogy between cardinal characteristics and highness properties of oracles

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What would a new open questions paper on randomness contain?

Randomness connecting to other fields (1)

- ...to computable analysis
  - A constructive bounded variation function is differentiable at each Martin-Löf random (Demuth, 1975; see upcoming BSL survey by Kučera, N and Porter).
  - Similar result holds for computable nondecreasing functions and computable randomness (Brattka, Miller, N, TAMS to appear).
  - Related result holds for weak Lebesgue points of L_1 computable functions, and Schnorr randomness (Pathak, 2009 ; Pathak, Rojas, Simpson 2014; Freer, Kjos-Hanssen, N, Stephan 2014).
- ... to reverse mathematics.
  - Implicit in the analysis papers: study the strength of the system asserting that relative to any oracle, there is a random.
  - A little of the connection is in Montalbán's BSL 2011 open questions paper on reverse math.

Randomness connecting to other fields (2)

- ... to ergodic theory.
  - Birkhoff's ergodic theorem for L_1-computable operator T and computable function f holds at each Martin-Löf random point (V'yugin 1997).
  - Lots of recent work (e.g. Gács, Hoyrup and Rojas 2009; Franklin and Towsner ta).
- ... to set theory.
  - Forcing with Borel sets of positive measure (Solovay 1970).
  - Randomness via effective descriptive set theory (Hjorth, N 2006, recent work by Greenberg, Monin).
  - Reimann and Slaman (2010) on "never continuously random"
  - Recent work of Kihara

Cardinal characteristics and highness properties

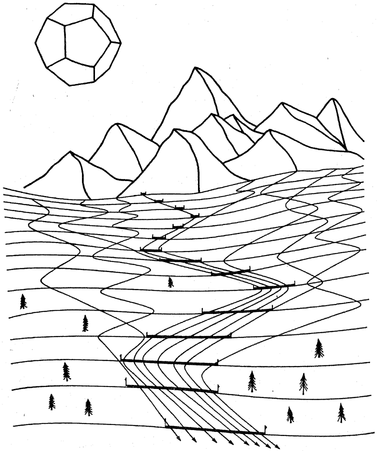
New interaction of set theory and computability/randomness:
A close analogy between cardinal characteristics of the continuum, and highness properties (indicating strength of a Turing oracle).

The unbounding number $\mathfrak{b}$ is the least size of a set of functions on the natural numbers so that no single function dominates them all. This corresponds to the usual highness: the oracle A computes a function that dominates all computable functions. This correspondence was first studied explicitly by Nicholas Rupprecht, a student of A. Blass (Arch. of Math. Logic, 2010).

Joint work with Jörg Brendle (Kobe), Andrew Brooke-Taylor (Bristol) and Selwyn Ng (NTU). (The first two may sound unfamiliar to you, because they are set theorists.)

Domination, and slaloms (Bartoszyński, 1987)

- For $f, g \in {}^\omega\omega$, let
  $f \leq^* g \Leftrightarrow f(n) \leq g(n)$
  for almost all n.
- A slalom is a function σ from ω the finite subsets of ω such that
  $$\forall n \, |\sigma(n)| \leq n^2.$$
- It traces f if
  $\forall^\infty n \, f(n) \in \sigma(n)$.



Picture in Bartoszyński's paper

Set theory versus computability

$\mathfrak{b} = \mathfrak{b}(\le^*)$: the least size of a set $F \subseteq {}^\omega\omega$ without a $\le^*$ upper bound.

there is $g \le_T A$ such that $f \le^* g$ for each computable f.

$\mathrm{cofin}(\mathcal{N})$: the least size of a collection of null sets covering all null sets.

A is not low for Schnorr tests.

$\mathfrak{d}(\in^*)$: the least size of a set of slaloms tracing all functions.

A is not computably traceable.

Thm. [Terw./Zamb. 2001]
Comp. traceable $\Leftrightarrow$ *low for Schnorr tests*

Thm. [Bartoszyński '84]
$\mathfrak{d}(\in^*) = \mathrm{cofin}(\mathcal{N})$.

Proofs of the Bartoszyński and Terwijn/Zambella theorems are similar (1)

Theorem (Bartoszyński '84)

$\mathfrak{d}(\in^*) = \mathrm{cofin}(\mathcal{N})$.

- Let $\mathcal{C}$ be the space of slaloms with the preorder of inclusion for almost every component.
- Let $\mathcal{N}$ be the null sets with inclusion.

To prove the harder inequality "$\leq$", one can use the following:

Lemma (Pawlikowski '85)

There is a function $\phi : \mathcal{C} \to \mathcal{N}$ such that the preimage of any set that is bounded above is again bounded above.

This relies on a coding of slaloms into sequences of open sets, and measure theoretic independence.

Proofs of the Bartoszyński and Terwijn/Zambella theorems are similar (2)

- trace is the computable version of slalom (we still have the condition $|\sigma(n)| \leq n^2|$).
- A is computably traceable if each $f \leq_{\mathrm{T}} A$ has a trace.
- A is low for Schnorr tests if each A-Schnorr null set is contained in a plain Schnorr null set.

Theorem (Terwjn and Zambella, 2001)

$A \subseteq \mathbb{N}$ is computably traceable $\Leftrightarrow$ A is low for Schnorr tests

Both implications use methods very similar to the ones above. In particular, the harder implication $(\Leftarrow)$ uses coding of functions into sequences open sets, and measure theoretic independence.

Unbounding and domination numbers of relations

Let $R \subseteq X \times Y$ be a relation. Let

$$\begin{aligned} \mathfrak{b}(R) &= \min\{|F| : F \subseteq X \wedge \forall y \in Y \exists x \in F \ \ \neg xRy\} \\ \mathfrak{d}(R) &= \min\{|G| : G \subseteq Y \wedge \forall x \in X \, \exists y \in G \ \ xRy\}. \end{aligned}$$

- $\mathfrak{b}(R)$ is called the unbounding number of R, and $\mathfrak{d}(R)$ the domination number.
- If R is a preordering without greatest element, then any set of covers is unbounded. So ZFC proves $\mathfrak{b}(R) \leq \mathfrak{d}(R)$.

Inclusion, membership in a class $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R})$ of small sets (such as null, meager)

Let $\subseteq_{\mathcal{S}}$ be inclusion on $\mathcal{S}$.

$$\begin{aligned} \mathrm{add}(\mathcal{S}) &:= \mathfrak{b}(\subseteq_{\mathcal{S}}) \\ \mathrm{cofin}(\mathcal{S}) &:= \mathfrak{d}(\subseteq_{\mathcal{S}}) \end{aligned}$$

Let $\in_{\mathcal{S}}$ be the membership relation on $\mathbb{R} \times \mathcal{S}$.

$$\begin{aligned} \mathrm{non}(\mathcal{S}) &:= \mathfrak{b}(\in_{\mathcal{S}}) = \min\{|U| : U \subseteq \mathbb{R} \wedge U \notin \mathcal{S}\} \\ \mathrm{cover}(\mathcal{S}) &:= \mathfrak{d}(\in_{\mathcal{S}}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{S} \wedge \bigcup \mathcal{F} = \mathbb{R}\} \end{aligned}$$

In the diagrams, going up or to the right means the cardinal gets bigger and ZFC knows it.

$$\begin{array}{ccc} \mathrm{non}(\mathcal{S}) & \longrightarrow & \mathrm{cofin}(\mathcal{S}) \\ \uparrow & & \uparrow \\ \mathrm{add}(\mathcal{S}) & \longrightarrow & \mathrm{cover}(\mathcal{S}) \end{array}$$

Combinatorial Cichoń diagram

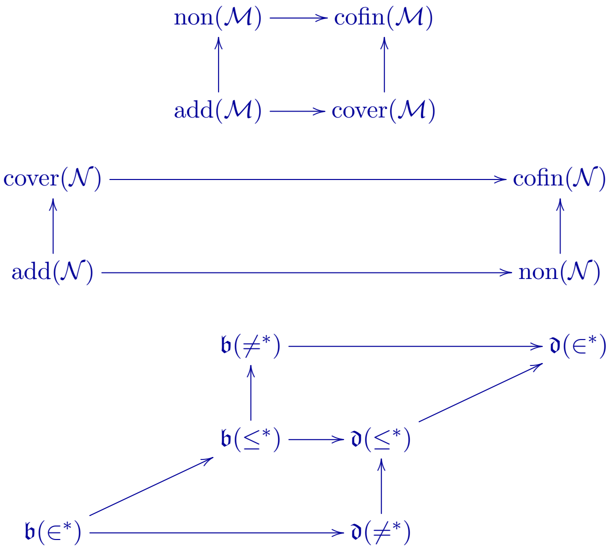
- $f \neq^* g \Leftrightarrow f(n) \neq g(n)$ for almost all n
- $f \leq^* g \Leftrightarrow f(n) \leq g(n)$ for almost all n
- $f \in^* \sigma \Leftrightarrow f(n) \in \sigma(n)$ for almost all n

ZFC knows that:

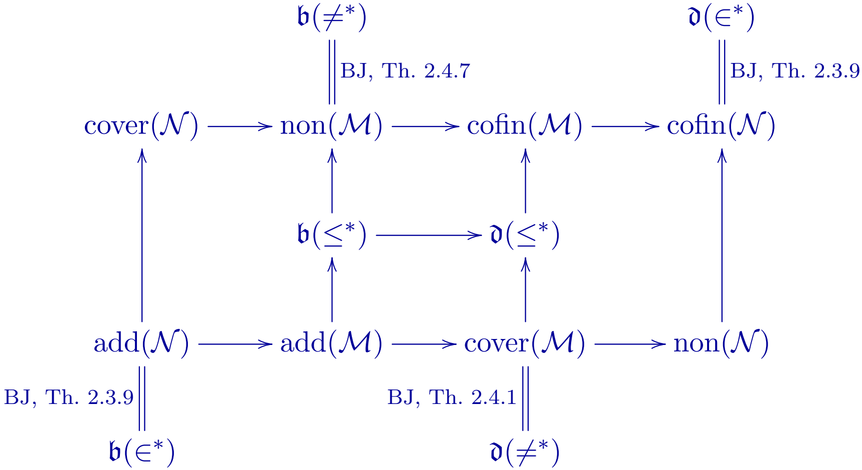
$$\begin{array}{ccc}
\mathfrak{b}(\neq^*) & \longrightarrow & \mathfrak{d}(\in^*) \\
\uparrow & & \uparrow \\
\mathfrak{b}(\leq^*) & \longrightarrow & \mathfrak{d}(\leq^*) \\
\uparrow & & \uparrow \\
\mathfrak{b}(\in^*) & \longrightarrow & \mathfrak{d}(\neq^*)
\end{array}$$

Joining three diagrams yields...

$\mathcal{M}$ = meager sets, $\mathcal{N}$ = null sets



… the (extended) Cichoń diagram of cardinals



All are in $[\omega_1, 2^{\aleph_0}]$. New arrows between $\mathcal{N}$, $\mathcal{M}$ due to Rothberger (1938); Pawlikowski (1985). BJ refers to Bartoszyński/Judah book. Each arrow can be made strict in a suitable model of ZFC.

Uniform transfer to the setting of computability (1)

Some of this was described in Rupprecht's thesis in a more informal way. Recall:

$$\begin{aligned}\mathfrak{b}(R) &= \min\{|F| : F \subseteq X \wedge \forall y \in Y \exists x \in F \neg xRy\}\\ \mathfrak{d}(R) &= \min\{|G| : G \subseteq Y \wedge \forall x \in X \exists y \in G\, xRy\}.\end{aligned}$$

Suppose we have specified what it means for objects x in X, y in Y to be computable in a Turing oracle A. Let the variable x range over X, and let y range over Y. We define the highness properties

$$\begin{aligned}\mathcal{B}(R) &= \{A : \exists y \leq_{\mathrm{T}} A\, \forall x \text{ computable } [xRy]\}\\ \mathcal{D}(R) &= \{A : \exists x \leq_{\mathrm{T}} A\, \forall y \text{ computable } [\neg xRy]\}.\end{aligned}$$

Note we are negating the set theoretic definitions. Reason: to "increase" a cardinal of the form $\min\{|F| : \phi(F)\}$, we need to introduce via forcing objects y so that $\phi(F)$ no longer holds in an extension model. This forcing corresponding to the construction of a powerful oracle computing a witness for $\neg\phi$.

Uniform transfer to the setting of computability (2)

Recall:

$$\begin{aligned} \mathcal{B}(R) &= \{A : \exists y \leq_{\mathrm{T}} A\, \forall x \text{ computable } [xRy]\} \\ \mathcal{D}(R) &= \{A : \exists x \leq_{\mathrm{T}} A\, \forall y \text{ computable } [\neg xRy]\}. \end{aligned}$$

- A Schnorr test is an effective sequence $(G_m)_{m \in \mathbb{N}}$ of Σ^0_1 sets such that each $\lambda G_m \leq 2^{-m}$ is a computable real uniformly in m.
- A set $\mathcal{F} \subseteq {}^\omega 2$ is Schnorr null if $\mathcal{F} \subseteq \bigcap_m G_m$ for a Schnorr test $(G_m)_{m \in \mathbb{N}}$.

$X = Y =$ class of null sets $\mathcal{N}$, R is inclusion $\subseteq_{\mathcal{N}}$.

- $\mathcal{B}(R)$ says that A computes a Schnorr test covering all plain Schnorr tests (“Schnorr engulfing”) and $\mathcal{D}(R)$ says that A is not low for Schnorr tests.

Basic diagram for null sets in computability

- $\mathcal{B}(\subseteq_{\mathcal{N}})$ says that A computes a Schnorr test covering all plain Schnorr tests ("Schnorr engulfing"), and $\mathcal{D}(\subseteq_{\mathcal{N}})$ says that A is not low for Schnorr tests.
- $\mathcal{D}(\in_{\mathcal{N}})$ says that A computes a Schnorr random, and $\mathcal{B}(\in_{\mathcal{N}})$ says that A computes a Schnorr test containing all computable reals ("weakly Schnorr engulfing", already studied by Rupprecht).

We obtain a diagram where upwards and right arrows mean implication of highness properties.

$$\begin{array}{ccc} \mathcal{D}(\in_{\mathcal{N}}) & \longrightarrow & \mathcal{D}(\subseteq_{\mathcal{N}}) \\ \uparrow & & \uparrow \\ \mathcal{B}(\subseteq_{\mathcal{N}}) & \longrightarrow & \mathcal{B}(\in_{\mathcal{N}}) \end{array}$$

Uniform transfer to the setting of computability (3)

Recall:

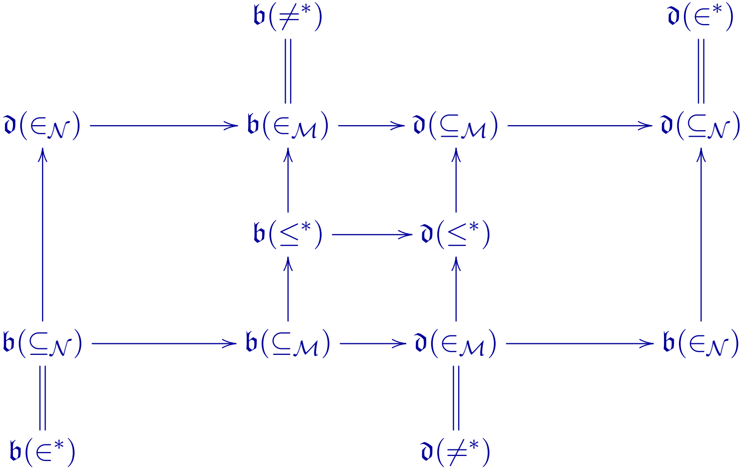
$$\begin{aligned} \mathcal{B}(R) &= \{A : \exists y \leq_{\mathrm{T}} A \, \forall x \text{ computable } [xRy]\} \\ \mathcal{D}(R) &= \{A : \exists x \leq_{\mathrm{T}} A \, \forall y \text{ computable } [\neg xRy]\}. \end{aligned}$$

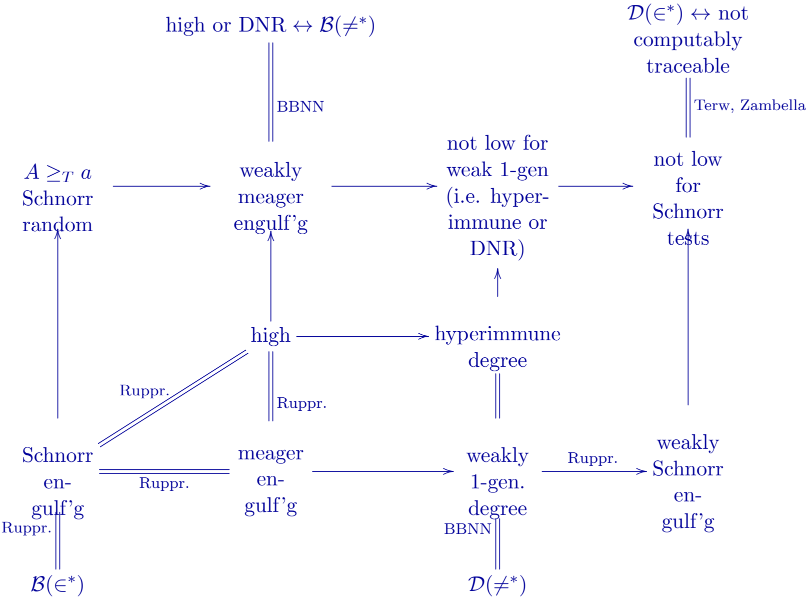
- An *effective* F_σ *class* has the form $\bigcup_m \mathcal{C}_m$, where the $\mathcal{C}_m$ are uniformly Π^0_1.
- A set $\mathcal{F} \subseteq {}^\omega 2$ is called effectively meager if it is contained in such a class $\bigcup_m \mathcal{C}_m$ where each $\mathcal{C}_m$ is nowhere dense.

Let $X = Y =$ meager sets, $R =$ inclusion.

- $\mathcal{B}(R)$ says that A computes a meager set covering all effectively meager sets ("meager engulfing")
- $\mathcal{D}(R)$ says that A is not low for meager sets.

The set-theoretic diagram in terms of unbounding/domination numbers




high or DNR ↔ $\mathcal{B}(\neq^*)$
BBNN
$\mathcal{D}(\in^*)$ ↔ not computably traceable
Terw, Zambella
$A \geq_T a$ Schnorr random
weakly meager engulf'g
not low for weak 1-gen (i.e. hyper-immune or DNR)
not low for Schnorr tests
high
hyperimmune degree
Ruppr.
Ruppr.
Schnorr en-gulf'g
Ruppr.
meager en-gulf'g
weakly 1-gen. degree
Ruppr.
weakly Schnorr en-gulf'g
Ruppr.
BBNN
$\mathcal{B}(\in^*)$
$\mathcal{D}(\neq^*)$

Meager engulfing

We discuss some new equalities in the diagram.

A is meager engulfing (Rupprecht) if A computes a meager set containing all plain meager sets. (This is the analog of $\mathrm{add}(\mathcal{M})$.)

Theorem (Rupprecht's thesis; short proof in BBNN)

A is high $\Leftrightarrow$ A is meager engulfing.

A is weakly meager engulfing if A computes a meager set containing all computable sets. (This is the analog of $\mathrm{non}(\mathcal{M})$.) We show equivalence with $\mathcal{B}(\neq^*)$.

Theorem (BBNN)

There is $f \leq_T A$ that eventually disagrees with each computable function $\Leftrightarrow$ A is weakly meager engulfing.

The property on the left is known to be equivalent to [high or DNR].

Further directions (1): set theory to computability

Study analogs of further cardinal characteristics.

- The *splitting number* $\mathfrak{s}$ is the least size of a subset $\mathcal{S}$ of $\mathcal{P}(\omega)$ such that every infinite set is split by a set in $\mathcal{S}$ into two infinite parts.
- The analog in computability theory: an infinite set $A \subseteq \omega$ is r-cohesive if it cannot be split into two infinite parts by a computable set.

Also: unreaping number and being of bi-immune free degree.

Further directions (2): set theory to computability

What happens when $x \leq_{\mathrm{T}} A$ is replaced by some other form of relative definability? For instance, by hyperarithmetical definability.

Disjoint work of Kihara and Monin shows that we have less collapsing.

- As before, every hyp-dominating is higher Schnorr engulfing.
- But hyperimmune is weaker than computing a weakly 1-generic.

How about arithmetical definability?

Understand the connection of forcing and constructions of powerful oracles.

Further directions (3): randomness to set theory

Is low for tests = low for randomness notion relevant in set theory?

Do some cardinal characteristics correspond to highness properties related to Martin-Löf randomness?

References

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- Nicholas Rupprecht. Effective correspondent to Cardinal characteristics in Cichoń's diagram. PhD Thesis, Univ of Michigan, 2010.

- These slides on my web page.