Metric spaces and computability theory

André Nies

SDF 60, Vienna

Handout version

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Abstract

We study similarity of Polish metric spaces. We consider the Scott rank, both forÊclassical and for continuous logic. The former is connected to isometry, the latter to having Gromov-Hausdorff distance zero. In the computable setting, we show that any two isometric compact metric spaces are Δ^0_3 isometric.

The various projects we review are joint with many researchers, among them Sy Friedman, Fokina and Koerwien; Ben Yaacov and Tsankov; Melnikov.

We study similarity relations for Polish metric spaces.

Determining isometries from the presentations

Theorem (with Melnikov)

Suppose M, N are isometric compact computable metric spaces. Then there is a Δ^0_3 isometry $g\colon M \to N$.
Sharpness: there are such M, N with no Δ^0_2 isometry.

Scott rank for Polish metric spaces: classical logic

Research direction (with Fokina, Friedman, Koerwien)

There are Polish metric spaces of arbitrarily high countable Scott rank.
Can the Scott rank be uncountable?

Scott rank for Polish metric spaces: continuous logic

Theorem (with Ben Yaacov, Tsankov)

The Scott rank w.r.t. continuous logic is countable.
The final Scott function between two spaces has value zero $\Leftrightarrow$ their Gromov-Hausdorff distance is zero (weaker than isometry).

Local approach to Polish metric spaces

Definition

- A Polish metric space $\mathcal{M}$ is a complete metric space (M, d) together with a dense sequence $(p_i)_{i \in \mathbb{N}}$. (This is actually a presentation of an abstract Polish metric space.)
- The space is computable if $d(p_i, p_k)$ is a computable real, uniformly in i, k.

Classic approach: only work with a few spaces at any time.

- Functional analysis: theorems only involve a few Banach spaces, such as $X, Y, X', Y', L(X, Y)$.
- We can use a fixed computable metric space as a setting for concepts from computability more general than Cantor Space. (For instance, Melnikov and N. study K-trivial points in computable metric spaces [Proc. AMS, 2013]).

Global approach to Polish metric spaces

Now we look at whole classes of Polish metric spaces, for instance all the compact ones.

Recall that a presentation is (M, d) together with a dense sequence $(p_i)_{i\in\mathbb{N}}$. All the presentations of Polish metric spaces together can be viewed as a closed set

$$\mathcal{P} \subseteq \mathbb{R}^{\omega\times\omega}.$$

This is sometimes called a hyperspace of Polish spaces;
see Su Gao, Invariant Descriptive Set Theory, Ch 14.
One studies equivalences on this hyperspace $\mathcal{P}$, such as isometry.

Compact spaces and categoricity

An internal way of understanding similarity

Given isometric presentations of metric spaces $\mathcal{M}, \mathcal{N}$,
can we determine an isometry from these presentations?

We are asking whether $\mathcal{M}, \mathcal{N}$ taken together "know" that they are isometric. This is not always the case, because isometry is Σ^1_1–complete.

Example (where we can internally determine an isometry)

- If computable metric spaces
$$\mathcal{M} = (M, d_M, (p_i)_{i\in\mathbb{N}}) \text{ and } \mathcal{N} = (N, d_N, (q_k)_{k\in\mathbb{N}})$$
are both isometric to $[0, 1]$, then there is a computable isometry g between them.
- To say that g is computable means that, on input a rational $\epsilon > 0$ and $i \in \mathbb{N}$, we can compute $k \in \mathbb{N}$ with $d_N(g(p_i), q_k) < \epsilon$.

Isometric compact spaces may fail to be Δ^0_2-isometric

Let α be a non-computable real with $\alpha = \sup_i r_i$ for $(r_i)_{i\in\mathbb{N}}$ a computable sequence of rationals. It is not hard to see that the natural presentations of the computable metric spaces

$$[0, \alpha] \text{ and } [-\alpha/2, \alpha/2]$$

are isometric, but not computably isometric.
(These presentations add whole closed intervals each time α increases.)

Can it be worse?

Theorem (Melnikov and N., 2013)

There are computable presentations L, R of a compact metric space so that no isometry is Δ^0_2.

The space is the closure of a computable sequence of elements $\sigma 0^\infty$ in Cantor space, for finite strings σ.

Can it be worse still?

Theorem (Melnikov and N., 2013; improved by J. Miller)

Suppose we have two computable presentations $(L, d_L, (p_i)_{i\in\mathbb{N}})$ and $(R, d_R, (q_k)_{k\in\mathbb{N}})$ of a compact metric space. Then there is a Δ^0_3 isometry $g\colon L \to R$.

Proof. Since any self-embedding of a compact metric space is onto, it suffices to obtain a Δ^0_3 embedding $g\colon\ L \to R$ (then use symmetry).

There is a Δ^0_2 function h such that $\{q_0, \dots, q_{h(n)}\}$ is a 2^{-n}-net for each n.

The $\Pi^0_1(\emptyset')$ tree T has at level n tuples in $\{q_0, \dots, q_{h(n)}\}^n$ which are possible isometric images of $\langle p_0, \dots, p_{n-1}\rangle$, up to an error of 2^{-n}.

With some compatibility condition from a level to the next, each infinite branch g of T gives rise to an isometric embedding

$$p_i \mapsto \lim_{n>i, n\to\infty} g(n)_i$$

(that is, map p_i to the limit of the i-th components of the tuples $g(n)$).

And the leftmost infinite branch is Δ^0_3; in fact, there is an infinite branch g with $g' \leq_{\mathrm{T}} \emptyset''$ by the low basis theorem. $\square$

A similar proof works for bi-Lipschitz equivalent presentations.

Scott analysis for Polish metric spaces

(classical logic)

α-equivalence of tuples in structures

Definition

Let M, N be $\mathcal{L}$-structures. Let $\bar{a}, \bar{b}$ be tuples of the same length from M, N.

- $\bar{a} \equiv_0 \bar{b}$ if the quantifier-free types of the tuples are the same.
- For a limit ordinal α, $\bar{a} \equiv_\alpha \bar{b}$ if $\bar{a} \equiv_\beta \bar{b}$ for all $\beta < \alpha$.
- $\bar{a} \equiv_{\alpha+1} \bar{b}$ if both of the following hold:
  - For all $x \in M$, there is $y \in N$ such that $\bar{a}\,x \equiv_\alpha \bar{b}\,y$
  - For all $y \in N$, there is $x \in M$ such that $\bar{a}\,x \equiv_\alpha \bar{b}\,y$

Back-and-forth systems and Scott rank

- A back-and-forth system for a pair of structures M, N is a set of finite partial isomorphisms with the extension property on both sides.
- $M \cong_p N$ if there is a nonempty back-and-forth system for the two structures.
- Suppose α is least such that $\equiv_\alpha$ implies $\equiv_{\alpha+1}$ for all tuples in M, N. If $\equiv_\alpha$ contains $\langle \emptyset, \emptyset \rangle$ then we get a non-empty back-and forth system, so $M \cong_p N$.
- For $M = N$, α is called the *Scott rank* of M.
  Note that this is $< |M|^+$.

For countable structures $\cong_p$ implies isomorphism. This can be used to define the Scott sentence, an $L_{\omega_1,\omega}$ sentence describing the structure within the countable structures.

Metric spaces as structures in first-order language

We view a metric space (X, d) as a structure for the signature

$$\{R_{<q}, R_{>q} \colon\ q \in \mathbb{Q}^+\},$$

where $R_{<q}$ and $R_{>q}$ are binary relation symbols.

- The intended meaning of $R_{<q}xy$ is that $d(x, y) < q$.
- The intended meaning of $R_{>q}xy$ is that $d(x, y) > q$.

Clearly, isomorphism is isometry.

Encouraging Fact

For Polish metric spaces A, B we have

$$A \cong_p B \Rightarrow A \text{ and } B \text{ are isometric.}$$

But, sorry, this doesn't mean there is a $L_{\omega_1,\omega}$ Scott sentence for A that works within the Polish metric spaces. In fact, as Alekos Kechris has kindly pointed out after the talk, there is no such sentence for every space, because this would give a classification of Polish metric spaces by countable structures.

Examples of Scott ranks

Natural spaces tend to have low Scott rank. For instance,

- Urysohn space $\mathbb{U}$ has Scott rank 0 (using that it is ultrahomogeneous.)
- A compact metric space has Scott rank at most ω (using an argument of Gromov that involves the distance relations).

Theorem (S. Friedman, Fokina, Koerwien, N.)

For each $\alpha < \omega_1$, there is a countable discrete ultrametric space M of Scott rank $\alpha \cdot \omega$.

M is given as the maximal branches on a subtree of $\omega^{<\omega}$. For $\sigma \neq \tau \in M$, the distance is 2^{-k} where k is the least disagreement.

Upper bounds on the Scott rank

The Scott rank of a Polish metric space (and in fact, any Borel structure) is less than the least ordinal α such that

$$L_\alpha(\mathbb{R}) \models \text{Kripke-Platek set theory}$$

(i.e., set theory with only Σ_1 replacement and Δ_0 comprehension).

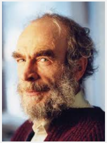
Question

Is the Scott rank of every Polish metric space countable?

Scott analysis for Polish metric spaces

(continuous logic)

The view of Mikhail Gromov



Gromov (1999 book on metrics and geometry, Ch. 3) says that isometry gives a boring category.
Instead, he studies similarity of Polish metric spaces M, N through the

Gromov–Hausdorff distance of M, N

the infimum of the Hausdorff distances of isometric embeddings of M, N into a third metric space.

- In general this can be 0 without the spaces being isometric.
- But distance 0 implies isometric for compact, and for discrete countable spaces.
- Hence, the GH-distance is not a Borel function (since isometry of countable discrete spaces is complete for S_∞ orbit relations).

Continuous Scott rank

Suppose A and B are metric spaces, $\bar{a} \in A, \bar{b} \in B$ tuples of the same length n. Let

$$r_{0,n}^{A,B}(\bar{a}, \bar{b}) = \min_i d(a'_i, b'_i),$$

where $a_i \mapsto a'_i, b_i \mapsto b'_i$ are isometric embeddings into a third metric space. (Note that this equals $\max_{i,k} |d(a_i, a_k) - d(b_i, b_k)|/2$; see Uspenskii 2008, Prop. 7.1.)

In the spirit of continuous logic, define by induction:

$$r_{\alpha+1,n}^{A,B}(\bar{a}, \bar{b}) = \max\left(\sup_{x \in A} \inf_{y \in B} r_{\alpha,n+1}^{A,B}(\bar{a}x, \bar{b}y), \sup_{y \in B} \inf_{x \in A} r_{\alpha,n+1}^{A,B}(\bar{a}x, \bar{b}y)\right)$$

$$r_{\alpha,n}^{A,B}(\bar{a}, \bar{b}) = \sup_{\beta < \alpha} r_{\beta,n}^{A,B}(\bar{a}, \bar{b}), \quad \text{for } \alpha \text{ limit.}$$

Fix A, B. Since the r_α are continuous, there is a countable least α with $r_\alpha(.,.) = r_{\alpha+1}(.,.)$ for each pair of tuples of the same length. Call this the continuous Scott rank of the pair A, B.

Characterizing GH-distance using Scott functions

Continuous Scott rank of A, B: least α with $r^{A,B}_{\alpha,n}(\bar{a},\bar{b}) = r^{A,B}_{\alpha+1,n}(\bar{a},\bar{b})$ for each n and each tuples $\bar{a},\bar{b}$ in A, B of length n.

The function at level α on empty tuples characterises the distance:

Theorem (Ben Yaacov, N., Tsankov)

Let α be the continuous Scott rank of A, B. Then the Gromov-Haussdorf distance of A, B equals $r^{A,B}_{\alpha,0}(\emptyset,\emptyset)$.

Let E_{GH} be the "near-isometry" relation that the Gromov-Hausdorff distance of Polish metric spaces A, B is 0. Using the above we can show that each equivalence class is Borel. (This is also known for isometry by Gao-Kechris.)

Some directions and open questions

- Develop effective categoricity for compact computable metric groups and other metric structures.
- Is homeomorphism for compact computable metric spaces Σ^1_1-complete in the sense of equivalence relations?
- Is the Scott rank of Polish metric spaces countable?
- Find a version of Lopez-Escobar for E_{GH}. Is every E_{GH}-invariant Borel set definable by a sentence in infinitary continuous logic?

References: slides on my web site; CiE 2013 paper by Melnikov/N.