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The complexity of similarity relations for Polish metric spaces

In October, André Nies gave a talk as part of the Universality and Homogeneity Trimester at the Hausdorff Institute for Mathematics in Bonn. The summary follows.

We are given a class of structures. We always mean concrete presentations of structures (rather than "up to isomorphism"). We address the following **leading questions:**

(a) Which similarity relations are there on the class?
(b) How complex are these similarity relations?
(c) If structures X, Y in the class are similar, how complex, relative to X, Y, is the means for showing this? For instance, if $X \cong Y$, can one compute and isomorphism from the structures?

For instance, in the model theoretic setting, we could be given the countable models of a first-order theory. Some answers to the questions in this setting are:

(a) isomorphism $\cong$, elementary equivalence $\equiv$, elementary equivalence $\equiv_\alpha$ for $L_{\omega_1,\omega}$ sentences of rank $< \alpha$.
(b) Isomorphism of countable graphs, linear orders, countable Boolean algebras is $\leq_B$ complete for orbit equivalence relations of continuous S_∞ actions ($\leq_B$ is Borel reducibility, S_∞ is the Polish group of permutations of ω).
(c) for $\cong$ is partially answered in computable model theory, with notions such as relatively computably categorical, where presentations of X, Y together uniformly compute an isomorphism if there is one at all. For instance, a dense linear order is r.c.c.

We will be mainly considering the **metric** setting. We have a class of Polish metric spaces. To answer (a): we have the following similarities which will be defined formally below.

Isometry $\cong_i$, homeomorphism $\cong_h$,

Gromov-Hausdorff distance 0, Lipschitz equivalence.

The former two are discussed in detail in [3, Ch. 14]. The latter two can be found in [4, Ch.3] (the first edition dates from 1998). After some preliminary facts, we will answer (b) and (c) for the metric setting. We also consider Polish metric spaces with some additional structure, such as Banach spaces, or spaces with a probability measure on the Borel sets.

Representing Polish metric spaces. We adopt the global view. Single structures are thought of as points in a "hyperspace". To endow this hyperspace with its own structure it matter how we represent the single structures. For metric spaces two ways are common.

(1) $\mathbb{U}$ denotes the Urysohn space, $F(\mathbb{U})$ its Effros algebra of closed subsets. Each Polish metric space is isometric to an element of $F(\mathbb{U})$. See [3, Ch. 14].
(2) A point $V = \langle v_{i.k} \rangle_{i,k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ is a *distance matrix* if V is a pseudometric on $\mathbb{N}$. Let M_V denote its completion. This means that in M_V we have a distinguished dense sequence of points $\langle p_i \rangle$ and present the space by giving their distances. We merely ask that V is a pseudometric in order to ensure that the set $\mathcal{M}$ of distance matrices is closed in $R^{\mathbb{N} \times \mathbb{N}}$.

Both representations are in a sense equivalent [3, Ch. 14]. However, the second one is better at describing the complexity of the space. For instance, a computable metric space $(M, d, \langle p_i \rangle)$ is given by a distance matrix w such that $w_{i,k} = d(p_i, p_k)$ is a computable real uniformly in i, k.

A Polish group action is a continuous action $G \times X \to X$ where G is a Polish group and X a Polish space. We write $G \curvearrowright X$. The orbit equivalence relation is $E_G^X = \{\langle x, y \rangle \colon \exists g \, [gx = y]\}$.

Polish metric spaces and the classical Scott analysis. A metric space (M, d) can be turned into a structure in the language with binary relations S_q for $q \in \mathbb{Q}^+$, where $S_q(a, b)$ holds if $d(a, b) < q$.

Definition 1.1. Let M be an $\mathcal{L}$-structure. We define inductively what it means for finite tuples of same length $\bar{a}, \bar{b}$ from M to be α-equivalent, denoted by $\bar{a} \equiv_\alpha \bar{b}$.

- $\bar{a} \equiv_0 \bar{b}$ if and only if the quantifier-free types of the tuples are the same.
- For a limit ordinal α, $\bar{a} \equiv_\alpha \bar{b}$ if and only if $\bar{a} \equiv_\beta \bar{b}$ for all $\beta < \alpha$.
- $\bar{a} \equiv_{\alpha+1} \bar{b}$ if and only if both of the following hold:
  - For all $x \in M$, there is some $y \in M$ such that $\bar{a}x \equiv_\alpha \bar{b}y$
  - For all $y \in M$, there is some $x \in M$ such that $\bar{a}x \equiv_\alpha \bar{b}y$

The *Scott rank* $\mathrm{sr}(M)$ of a structure M is defined as the smallest α such that $\equiv_\alpha$ implies $\equiv_{\alpha+1}$ for all tuples of that structure. We remark that always $\mathrm{sr}(M) < |M|^+$.

Fact 1.2. *A Polish space has Scott rank* 0 *iff it is ultrahomogeneous.*

Friedman, Körwien and Nies (2012) showed that for each $\alpha < \omega_1$, there is an countable Polish ultrametric space M such that $\mathrm{sr}(M) = \alpha \times \omega$.

Question 1.3.
(a) Does every Polish metric space have countable Scott rank?
(b) Can it in fact be described within the class of Polish metric spaces by an $L_{\omega_1,\omega}$ sentence?

I. Isometry $\cong_i$. In 1998 Anatoly Vershik [6] asked about the complexity of isometry $\cong_i$ on Polish metric spaces, and in particular if one can assign invariants. The answer was a resounding no.

Theorem 1.4 (Gao-Kechris 2000; Clemens; see [3], Ch. 14)**.** $\cong_i \leq_B E^{F(\mathbb{U}}_{Iso(\mathbb{U})}$.
For every $E = E^X_G$ we have $E \leq_B \cong_i$.

Let $\mathcal{K}$ be the class of compact metric spaces. Note that this is Π^0_3 with respect to the distance matrix representation of Polish metric spaces, because compactness is equivalent to being totally bounded. Isometry of compact spaces is much simpler: points in some fixed Polish space serve as invariants.

Theorem 1.5 (Essentially Gromov [4], Thm 3.27,5)**.**

$$\cong_i \cap (\mathcal{K} \times \mathcal{K}) \leq_B id_{\mathbb{R}}.$$

Proof. Gromov shows that the sequence of sets of $n \times n$ distance matrices that occur in a compact space X constitute a complete set of invariants. Each such matrix is a point in a compact set $K_n(X) \subseteq \mathbb{R}^{n^2}$. The sequence of such compact sets can be represented by a single point in a Polish space, say $\mathbb{R}$. □

Computable versions. The distance matrices form an effectively closed set, and thus can be coded as the infinite branches of a Π^0_1 tree $\subseteq 2^{<\omega}$. Such a branch provides data of the form $|v_{i,k} - q| < \epsilon$ for $q \in \mathbb{Q}^+_0$, $\epsilon \in \mathbb{Q}^+$.

Let V_e denote the e-th computable distance matrix. The domain of this partial computable function grows as long as the data are consistent with being a distance matrix; if seen to be not (a Σ^0_1 event) it stops. Being total is Π^0_2.

Let M_e denote the computable metric space given by the e-th (total) distance matrix V_e.

Proposition 1.6 (Fokina et al. [2])**.** *$\{\langle e, k\rangle \colon\ M_e \cong_i M_k\}$ is a Σ^1_1 complete equivalence relation.*

Proposition 1.7. *The set C of indices for compact computable metric spaces is Π^0_3. Isometry is Π^0_2 within that set, that is, of the form $E \cap C \times C$ where E is Π^0_2 relation.*

II. Having Gromov-Hausdorff distance 0**.** The following is ongoing work with Itai Ben Yaacov and Todor Tsankov. One thinks of two metric spaces X, Y as isometric within error ϵ if they can be isometrically embedded into a third metric space Z in such a way that the usual Hausdorff distance of the two images is at most ϵ. "X, Y isometric within error 0" then means that the completions of X, Y are isometric. We let the Gromov- Hausdorff distance be

$$d_{GH}(X, Y) = \inf\{\epsilon \colon\ X, Y \text{ isometric within error } \epsilon\}.$$

For instance, if we let $X = \{0, 1\}$ and $Y = \{1/4, 3/4\}$, then

$$d_{GH}(X, Y) = 1/4.$$

So, are there examples of non-isometric spaces X, Y with GH-distance 0? Neither space can be compact (Gromov). Also there is no positive lower bound on the distance of distinct points, otherwise a near isometry with error less than that bound will be an isometry. During the HIM talk, Nies mentioned an example: let $\mathbb{E}$ be the unit sphere of the Gurarij space. Let $v \in \mathbb{E}$ be smooth, and w be non-smooth. Let $X = Y = \mathbb{E} \cup a, b$, with $d_X(a,b) = d_Y(a,b) = 3$. We set $d_X(v,a) = d_X(v,b) = 3$, and $d_Y(w,a) = d_Y(w,b) = 3$. Any isometry would have to map v to w, which is impossible. However, by general properties of the Gurarij space, $d_{GH}(X,Y) = 0$.

After the talk Matatiahou Rubin and Philipp Schlicht constructed further, in a sense simpler examples. See Appendix 1.

Bi-Katetov functions. One can describe being isometric within error ϵ without referring to a third space. A *bi-Katetov function* $f\colon X \times Y \to \mathbb{R}$ is defined as

$$f(x,y) = d_Z(i(x), j(y)),$$

where i, j are embeddings into some metric space as above. Equivalently, f is 1-Lipschitz in both variables and

$$d_A(x,w) \le f(x,y) + f(w,y)$$
$$d_B(y,z) \le f(x,y) + f(x,z)$$

A bi-Katetov function f can be seen as an approximate isometry. Its error q_f is given by

$$q_f = \max(\sup_x \inf_y f(x,y), \sup_y \inf_x f(x,y)).$$

By definition this equals the Hausdorff distance of the isometric images above.

For instance, if there is an actual onto isometry $\theta : X \to Y$, we can let $f(x,y) = d_Y(\theta(x), y)$ and obtain the least possible error 0. Conversely, as mentioned above, if the spaces are complete and the error is 0 then there is an onto isometry.

Clearly we have

$$d_{GH}(X,Y) = \inf_f q_f,$$

where f runs through all the bi-Katetov functions on $X \times Y$.

Continuous Scott analysis. We define approximations to d_{GH} from below by induction on countable ordinals.

Suppose $\bar{a} = \langle a_i \rangle_{i<n}$ and $\bar{b} = \langle b_i \rangle_{i<n}$ are enumerated finite metric spaces. Following [?] define

$$r_{0,n}(\bar{a}, \bar{b}) = \inf_{f \text{ is bi-Katetov on } \bar{a}\times\bar{b}} \max_{i<n} f(a_i, b_i).$$

An explicit expression for this is given by [?]*Proposition 7.1:

$$r_{0,n}(\bar{a}, \bar{b}) = \varepsilon/2 \ \text{ where } \ \varepsilon = \min_{i,k<n} |d(a_i, a_k) - d(b_i, b_k)|. \tag{1}$$

(In fact, Uspenskii builds a bi-Katetov function such that $f(a_i, b_i) = \varepsilon/2$ for each i.)

Definition 1.8. Suppose A and B are metric spaces and $\bar{a} \in A^n, \bar{b} \in B^n$. Define by induction on ordinals α:

$$r_{0,n}^{A,B}(\bar{a},\bar{b}) = r_{0,n}(\bar{a},\bar{b})$$

$$r_{\alpha+1,n}^{A,B}(\bar{a},\bar{b}) = \max\big(\sup_{x\in A}\inf_{y\in B} r_{\alpha,n+1}^{A,B}(\bar{a}x,\bar{b}y), \sup_{y\in B}\inf_{x\in A} r_{\alpha,n+1}^{A,B}(\bar{a}x,\bar{b}y)\big)$$

$$r_{\alpha,n}^{A,B}(\bar{a},\bar{b}) = \sup_{\beta<\alpha} r_{\beta,n}^{A,B}(\bar{a},\bar{b}), \quad \text{for } \alpha \text{ a limit ordinal.}$$

Given a metric space (X,d) and $n \geq 1$, we equip X^n with the "maximum" metric $d(\bar{u},\bar{v}) = \max_{i<n} d(u_i,v_i)$. The following are not hard to check.

Lemma 1.9. *Fix separable metric spaces A, B of finite diameter.*

(1) *For each α and each n, the functions $r_{\alpha,n}^{A,B}(\bar{a},\bar{b})$ are 1-Lipschitz in $\bar{a}$ and $\bar{b}$.*
(2) *The functions $r_{\alpha,n}^{A,B}(\bar{a},\bar{b})$ are nondecreasing in α.*
(3) *There is $\alpha < \omega_1$ after which all the $r_{\alpha,n}^{A,B}$ stabilize.*

Theorem 1.10 (Ben Yaacov, Nies, Tsankov 2013)**.** *Let A, B be separable metric spaces of finite diameter. Let α^* be such that $r_{\alpha^*+1,n}^{A,B} = r_{\alpha^*,n}^{A,B}$ for each n. Then*

$$r_{\alpha^*,0}^{A,B} = d_{GH}(A,B).$$

For a proof see appendix 2.

Definition 1.11. The *continuous Scott rank* of A is least α for which

$$r_{\alpha,n}^{A,A}(\bar{a}_1,\bar{a}_2) = r_{\alpha+1,n}^{A,A}(\bar{a}_1,\bar{a}_2), \quad \text{for all } n, \bar{a}_1, \bar{a}_2 \in A^n.$$

One can define an equivalence relation E_{GH} on the set of distance matrices $\mathcal{M}$ by

$$A E_{GH} B \iff d_{GH}(A,B) = 0.$$

Using the continuous Scott analysis we can show:

Theorem 1.12. *Each E_{GH} class is Borel.*

III. Homeomorphism $\cong_h$. We collect some results, most of which are proved in [3, Ch. 14]. For general Polish metric spaces, $\cong_h$ is merely known to be $\mathbf{\Sigma^1_2}$. Homeomorphism of compact metric spaces X, Y is analytic, because homeomorphisms are uniformly continuous. In fact, by the Banach-Stone theorem, we have

$$X \cong_h Y \Leftrightarrow \mathcal{C}(X) \cong_i \mathcal{C}(Y);$$

so by the aforementioned results of Gao and Kechris on isometry [**?**], $\cong_h$ on compact metric spaces is Borel reducible to an orbit equivalence relation. (A similar argument works for locally compact metric spaces, using $\mathcal{C}_0(X)$, the C^* algebra of continuous functions vanishing at ∞; however, for a Polish metric space, to be locally compact is known to be properly $\mathbf{\Pi}^1_1$.)

Camerlo and Gao [1] proved that graph isomorphism is Borel reducible to homeomorphism of totally disconnected compact metric spaces (i.e., separable Stone spaces). One notes that countable compact metric spaces X won't work here, because X is scattered and hence given by the Cantor-Bendixson rank α together the size of the last set $X^{(\alpha)}$.

The main question remains open.

Question 1.13. *Determine the complexity with respect to $\leq_B$ of $\cong_h$ on compact metric spaces.*

In contrast, in the computable case the complexity is known to be as large as possible.

Theorem 1.14. *Homeomorphism of compact computable metric spaces is complete with respect to computable reductions for Σ^1_1 equivalence relations on ω.*

Proof. Friedman et al. [2] showed this for isomorphism of computable graphs. It can be verified that the construction Camerlo-Gao use for providing their Borel reduction is effective. Hence, if the given graph is computable, then uniformly in its index they build a compact computable metric space. □

The complexity of particular isometries. Let us return to the leading questions posed initially. It appears that Questions (b) and (c) are closely connected:

It is easy to detect that X is similar to Y $\Leftrightarrow$
we can determine from X, Y a means via which the similarity holds.

We will provide some evidence for this thesis, first for compact metric spaces, and then for metric measure spaces studied by Gromov [4] and Vershik. For a function g, let g' be the halting problem relative to the graph of g.

Theorem 1.15 (Melnikov, Nies [5])**.** *Let X, Y be compact metric spaces. Let A be an oracle Turing equivalent to the Turing jump of (the presentation of) X together with Y.*

(a) *If $X \cong_i Y$ then there is an isometry g such that $g' \leq_{\mathrm{T}} A''$.*
(b) *there a isometric compact computable metric spaces X, Y with no isometry $g \leq_{\mathrm{T}} \emptyset'$.*

For (a) note that is suffices to build g an isometric embedding. By compactness we can view embeddings as branches on a subtree $T \subseteq \omega^{<\omega}$ with an A' computable bound on the level size. Now apply the low basis theorem relative to A' in order to obtain g.

A metric measure (m.m.) space has the form $T = (X, \mu, d)$ where (X, d) is Polish, μ a Borel probability measure. We may assume that $\mu U > 0$ for any non-empty open U; otherwise, replace X by the least conull closed set.

Theorem 1.16 (Gromov (1997), see [4])**.** *Measure-preserving isometry of m.m. spaces is smooth.*

Gromov used as invariants the sequence of distributions D_n of the distance matrix of n randomly chosen points. He used moments to show that $\langle D_n \rangle_{n \in \mathbb{N}}$ is a complete invariant for the m.m. space T.

In 1996 Vershik [6] also gave a proof; also see the survey [7]. He describes T by the single invariant D_T, the distribution of the distance matrix of a randomly chosen infinite sequence (x_i). More formally, D_T is the pushout measure of $d(x_i, x_k)$ on the space $\mathcal{M} \subseteq \mathbb{R}^{\omega \times \omega}$ of distance matrices. He used a form of the law of large numbers to reconstruct T from D_T.

We give an effective analysis of Vershik's proof. Let $\mathcal{O} \subseteq \omega$ be some Π^1_1 complete set.

Theorem 1.17. *Suppose T_1, T_2 are computable m.m. spaces (the measure of Boolean combinations of balls is uniformly computable). Then there is a measure-preserving isometry Θ such that $\Theta \leq_{\mathrm{T}} \mathcal{O}^{(\alpha)}$ for some computable ordinal α.*

It remains to be determined whether this is necessarily bad, or if the complexity of a Θ can be lowered.

Appendix 1: Non-isometric discrete spaces with GH distance 0. M. Rubin gave an example of two non-isometric Banach spaces at distance 0: let $D = \langle p_i \rangle_{i \in \mathbb{N}}$ be a dense sequence of distinct elements in $(1, 2)$, say. Let U_p be the 2-dimensional $\mathbb{R}$ vector space with ℓ_p norm. Let E_D be the c_0-sum of the spaces U_{p_i}. That is, null sequences, with norm the maximum of the individual ℓ_{p_i} norms. If we have two dense sequences with different sets of members, the spaces are at distance 0, but not isometric.

The following result of Schlicht and Rubin shows that there is a single E_{GH} class such that the isometry equivalence relation inside is Borel bi-reducible with identity on $2^{\mathbb{N}}$. In particular, there are continuum many non-isometric spaces with discrete topology that are mutually at GH distance 0.

We equip $[0, \epsilon] \times (\omega + 1) \times \mathbb{R}$ with the metric d defined by $d((x, i, y), (x', i', y')) = 1$ if $(x, i) \neq (x', i')$ and $d((x, i, y), (x', i', y')) = |y - y'|$ if $(x, i) = (x', i')$.

Definition 1.18. Suppose that $f \colon [0, \epsilon] \to \omega + 1$ is a function.

(1) Let $X_f = \{(x, i, 0), (x, i, x) \in [0, \epsilon] \times (\omega + 1) \times \mathbb{R} \mid i \leq f(x)\}$ with the metric from $[0, \epsilon] \times (\omega + 1) \times \mathbb{R}$.
(2) Let $\mathrm{supp}(f) = \{x \in [0, \epsilon] \mid f(x) \neq 0\}$ denote the *support* of f.
(3) Let $\mathrm{bound}(f) = \{(x, i) \mid x \in \mathrm{supp}(f),\ i \leq f(x)\}$.

If $|\mathrm{supp}(f)| = \omega$, then X_f is a discrete countable metric space with distance set $\mathrm{supp}(f) \cup \{0, 1\}$.

Proposition 1.19. *Suppose that $\epsilon \leq \frac{1}{2}$. Suppose that $f_0 \colon [0, \epsilon] \to \omega + 1$ is a function such that $\mathrm{supp}(f_0)$ is a countable dense subset of $[0, \epsilon]$. Then $\mathrm{id}_{^{\omega}2}$ is Borel reducible to $\mathrm{Iso} \restriction [X_{f_0}]_{GH}$.*

Proof. Note that for arbitrary functions $f, g \colon [0, \epsilon] \to \omega + 1$, X_f, X_g are isometric if and only if $f = g$.

Claim 1.20. *Suppose that $f, g \colon [0, \epsilon] \to \omega + 1$ are functions such that $\mathrm{supp}(f)$, $\mathrm{supp}(g)$ are countable dense subsets of $[0, \epsilon]$. Then $d_{GH}(X_f, X_g) = 0$.*

Proof. Note that for every $\delta > 0$, there is a bijection $h \colon \mathrm{bound}(f) \to \mathrm{bound}(g)$ such that $|x - h(x, i)_0| < \delta$ for all $(x, i) \in \mathrm{bound}(f)$. Let $h \times \mathrm{id} \colon X_f \to \mathrm{bound}(g) \times \mathbb{R}$, $(h \times \mathrm{id})(x, i, y) = (h(x, i), y)$. Then $h \times \mathrm{id}$ is distance preserving and $d_H((h \times \mathrm{id})[X_f], X_g) \leq \delta$. Hence $d_{GH}(X_f, X_g) \leq \delta$. $\square$

Let $D_q = \{0, q\}$ for $q > 0$. Suppose that $(q_n, i_n)_{n \in \omega}$ is an enumeration of $\mathrm{bound}(f_0)$ without repetitions. Suppose that X is a complete metric space with $d_{GH}(X, X_{f_0}) = 0$. Suppose that $0 < \delta < 1$. Since $d_{GH}(X, X_{f_0}) < \frac{\delta}{3}$, X is of the form $X = \bigsqcup_{n \in \omega} X_n^\delta$ with

(1) $d_{GH}(X_n^\delta, D_{q_n}) < \delta$ and
(2) $|d(x, y) - 1| < \delta$ if $x \in X_m^\delta$, $y \in X_n^\delta$, and $m \neq n$.

Let $X_n = X_n^{\frac{1}{2}}$. Conditions 1 and 2 imply that for all $\delta < \frac{1}{2}$ and all n, there is some m with $X_m^\delta = X_n$. Hence for each n there is a sequence $(n_i)_{i\in\omega}$ in ω with $d_{GH}(X_n, D_{q_{n_i}}) < \frac{1}{2^i}$. It follows that $1 \leq |X_n| \leq 2$. Let $p_n = d(x,y)$ if $X_n = \{x,y\}$. Let $A = \{p_n \mid n \in \omega\}$.

Claim 1.21. *$d(x,y) = 1$ for all $x \in X_m$ and $y \in X_n$ with $m \neq n$.*

Proof. This follows from Condition 2 and since for all $\delta < \frac{1}{2}$ and all k, there is some l with $X_l^\delta = X_k$. □

Claim 1.22. *$A \subseteq [0,\epsilon]$.*

Proof. Suppose that $X_n = \{x,y\}$ and $\eta = d(x,y) - \epsilon > 0$. Suppose that $X_n = X_m^\eta$. This contradicts the fact that $d_{GH}(X_m^\eta, D_{q_m}) < \eta$ by Condition 1. □

Claim 1.23. *A is dense in $(0,\epsilon)$.*

Proof. Suppose that $U \subseteq (0,\epsilon)$ is nonempty and open with $U \cap A = \emptyset$. Suppose that $(q_n - \delta, q_n + \delta) \subseteq U$. This contradicts the fact that $d_{GH}(X_n^{\frac{\delta}{2}}, D_{q_n}) < \frac{\delta}{2}$ by Condition 1. □

Let $f\colon [0,\epsilon] \to \omega+1$, $f(x) = 0$ if $x \notin A$, $f(0) = i$ if $|\{n \in \omega \mid |X_n| = 1\}| = i$, and $f(z) = i$ if $|\{n \in \omega \mid \exists x,y \in X_n\ d(x,y) = z\}| = i$ for $z \in (0,\epsilon]$. Then X_f, X are isometric. □

Appendix 2: Proof of Theorem 1.10. Since A, B are fixed we suppress them in our notations. Variables a, a_i etc range over A, and b_i etc. range over B. For tuples $\bar a, \bar b$ of length k, let

$$\delta_k(\bar a, \bar b) = \inf_f \{\max(q_f, \max_{i<k} f(a_i, b_i))\},$$

where f ranges over bi-Katetov functions on $A \times B$. We show that for each n and tuples $\bar a, \bar b$ of length n,

$$r_{\alpha^*,n}(\bar a, \bar b) = \delta_n(\bar a, \bar b).$$

For $n = 0$ this establishes the theorem.

Firstly, we show by induction on ordinals α that

$$r_{\alpha,n}(\bar a, \bar b) \leq \delta_n(\bar a, \bar b)).$$

The cases $\alpha = 0$ and α limit ordinal are immediate. For the successor case, suppose that $\delta_n(\bar a, \bar b) < s$ via a bi-Katetov function f on $A \times B$. For each $x \in A$ we can pick $y \in B$ such that $f(x,y) < s$. Then $\delta_{n+1}(\bar a x, \bar b y) < s$ via the same f. Inductively we have $r_{\alpha,n+1}(\bar a x, \bar b y) < s$. Similarly, for each $y \in B$ we can pick $x \in A$ such that $r_{\alpha,n+1}(\bar a x, \bar b y) < s$. This shows that $r_{\alpha,n}(\bar a, \bar b) \leq s$.

Secondly, we verify that

$$\delta_n(\bar a, \bar b)) \leq r_{\alpha^*,n}(\bar a, \bar b)$$

Let $r_{\alpha^*,n}(\bar a, \bar b) < t$. We combine a back-and-forth argument with the compactness of the space of bi-Katetov functions in order to build a bi-Katetov function f with $q_f \leq t$ and $\max_{i<n} f(a_i, b_i) \leq t$.

To do so we extend $\bar{a}, \bar{b}$ to dense sequences in A, B respectively. Let $D \subseteq A, E \subseteq B$ be countable dense sets. Let $\bar{u}^k$ denote a tuple of length k; in particular, we can write $\bar{a} = \bar{a}^n$ and $\bar{b} = \bar{b}^n$. We ensure that

$$r_{\alpha^*,k}(\bar{a}^k, \bar{b}^k) < t \text{ for each } k \geq n.$$

Suppose $\bar{a}^k, \bar{b}^k$ have been defined. If k is even, let a_k be the next element in D. Using $r_{\alpha^*+1,k}(\bar{a}^k, \bar{b}^k) = r_{\alpha^*,k}(\bar{a}^k, \bar{b}^k)$ we can choose b_k so that $r_{\alpha^*,k+1}(\bar{a}^{k+1}, \bar{b}^{k+1}) < t$. Similarly, if k is odd, let b_k be the next element in E and choose a_k as required.

By Lemma 1.9(2) we have $r_{0,k}(\bar{a}^k, \bar{b}^k) < t$ for each $k \geq n$ via some bi-Katetov function $\widetilde{f_k}$ defined on $\{a_0, \ldots, a_{k-1}\} \times \{b_0, \ldots, b_{k-1}\}$. We can extend this to a bi-Katetov function f_k defined on $A \times B$ as noted in Remark **??**. By the compactness of the space of bi-Katetov functions on $A \times B$, viewed as elements of $\mathbb{R}^{D \times E}$, there is a subsequence $k_0 < k_1 < \ldots$ such that $\langle f_{k_u} \rangle$ converges pointwise to a bi-Katetov function f. Since bi-Katetov functions are 1-Lipschitz in both arguments, this implies $\lim_u f_{k_u}(a_p, b_p) = f(a_p, b_p)$ for each p. Therefore $f(a_p, b_p) \leq t$. This implies $q_f \leq t$ as required.

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