Categoricity for compact computable metric spaces

Alexander Melnikov and André Nies

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THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Synopsis

We study the complexity of isometry between compact computable metric spaces.

Index sets:

The class of computable indices for compact metric spaces is Π^0_3.
Being isometric is Π^0_2 within that class.

Determining isometries from the presentations:

Theorem

- Suppose M, N are isometric compact computable metric spaces. Then there is an isometry $g \colon M \to N$ such that $g' \leq_T \emptyset''$.
- There are such M, N where $\emptyset'$ cannot compute an isometry.

Local approach to Polish metric spaces

Definition

- A Polish metric space $\mathcal{M}$ is a complete metric space (M, d) together with a dense sequence $(p_i)_{i \in \mathbb{N}}$.
- The space is computable if $d(p_i, p_k)$ is a computable real, uniformly in i, k.

Classic approach: only work with a few spaces at any time.

- Functional analysis: theorems only involve a few Banach spaces, such as $X, Y, X', Y', L(X, Y)$.
- We can use a fixed computable metric space as a setting for concepts from computability more general than Cantor Space.

  – For instance, one defines a point $x \in M$ to be computable if there is computable function f such that $x = \lim_k p_{f(k)}$ and $d(x, p_k) \leq 2^{-k}$.

  – Melnikov and N. study K-trivial points in computable metric spaces [Proc. AMS, 2013]. Their main result is that each such point has a K-trivial Cauchy name $(p_{f(i)})_{i \in \mathbb{N}}$, namely, $K(f\restriction_n) \leq K(n) + O(1)$. This implies that K-triviality is closed under computable maps.

Global approach to Polish metric spaces

Now we look at whole classes of Polish metric spaces.

- The representations of Polish metric spaces form a closed set $\mathcal{P} \subseteq \mathbb{R}^{\omega \times \omega}$.
  (This is sometimes called a hyperspace of Polish spaces;
  see Su Gao, Invariant Descriptive Set Theory, Ch 14.)
- The Polish metric spaces with diameter bounded by 1 form a computable metric subspace of the Hilbert cube $[0,1]^{\omega \times \omega}$; the computable spaces are the computable points.

- One studies equivalences on $\mathcal{P}$, such as isometry.
- we can measure similarity of $M, N \in \mathcal{P}$ by the

  Gromov–Hausdorff distance of M, N

  (this is the infimum of the Hausdorff distances of isometric embeddings of M, N into a third metric space).

The viewpoint of invariant descriptive set theory

Let $\mathcal{X}, \mathcal{Y}$ be Polish topological spaces. Let E be an equivalence relation on $\mathcal{X}$. Let F be an equivalence relation on $\mathcal{Y}$.

We write $E \leq_B F$ if there is a Borel map $\phi \colon \mathcal{X} \to \mathcal{Y}$ such that

$$u \, E \, v \Leftrightarrow \phi(u) \, F \, \phi(v).$$

That is, E is Borel many-one reducible to F.

Theorem (Gao-Kechris 2003/Clemens)

Let E be equivalence relation arising from a Polish group action on $\mathcal{X}$. Then $E \ \leq_B \cong_i$, where $\cong_i$ denotes isometry of Polish metric spaces.

Isometry of compact metric spaces is much simpler

Definition (Distance relations in metric spaces)

For a metric space $\mathcal{X}$, let $D_n(\mathcal{X})$ denote the set of all $n \times n$ matrices $d(x_i, x_k)_{i,k<n}$, where $x_0, \ldots, x_{n-1} \in \mathcal{X}$.

Theorem (Gromov, 1999; see Su Gao, 2009)

Let $\mathcal{X}_0, \mathcal{X}_1$ be compact metric spaces such that $D_n(\mathcal{X}_0) = D_n(\mathcal{X}_1)$ for each n. Then $\mathcal{X}_0$ and $\mathcal{X}_1$ are isometric.

- The sequence of compact sets $D_n(\mathcal{X}) \subseteq \mathbb{R}^n$ can be encoded by a single point in a standard Polish space.
- So, $\cong_i$ on compact spaces is *smooth*, that is, Borel reducible to the identity on $\mathbb{R}$.

The same in the computable setting

Theorem (Follows from Fokina et al., 2010)

Isometry of computable metric spaces is (computably) many-one complete for Σ^1_1 equivalence relations.

Fokina et al. actually show Σ^1_1-completeness for isomorphism of countable graphs. These graphs can be coded into countable ultrametric spaces.

Proposition

The class of computable indices e for compact metric spaces $\mathcal{M}_e$ is Π^0_3. Being isometric is Π^0_2 within that class.

Being Π^0_3 follows from the definition of compactness via ϵ-nets.
Isometry is Π^0_2 by Gromov's argument: for compact spaces,

$$\text{“}\forall n\, [D_n(\mathcal{M}_e) = D_n(\mathcal{M}_i)]\text{”}$$

is a Π^0_2 predicate of e and i. It is not Σ^0_2.
There is no many-one complete Π^0_2 equivalence relation by Ianovski, Miller, Ng and N., “Complexity of equivalence relations and preorders from computability theory”, submitted 2013.

An internal way of understanding similarity

Given isometric presentations of metric spaces $\mathcal{M}, \mathcal{N}$,
can we determine an isometry from these presentations?

We are asking whether $\mathcal{M}, \mathcal{N}$ taken together "know" that they are isometric. This is not always the case, because isometry is Σ^1_1–complete.

Example

- If computable metric spaces
$$\mathcal{M} = (M, d_M, (p_i)_{i\in\mathbb{N}}) \text{ and } \mathcal{N} = (N, d_N, (q_k)_{k\in\mathbb{N}})$$
are both isometric to $[0,1]$, then there is a computable isometry g between them.
- To say that g is computable means that from a rational $\epsilon > 0$ and $i \in \mathbb{N}$, we can compute $k \in \mathbb{N}$ with $d_N(g(p_i), q_k) < \epsilon$.

- Such a computable metric space is called computably categorical.
- ℓ_2 and Urysohn space $\mathbb{U}$ are computably categorical (Melnikov).
- ℓ_1 isn't (Pour-El/Richards); $\mathcal{C}[0,1])$ isn't (Melnikov).

Isometric compact spaces may fail to be Δ^0_2-isometric

Let α be a left-r.e. non-computable real. It is not hard to see that the natural presentations of the computable metric spaces

$$[0, \alpha] \text{ and } [-\alpha/2, \alpha/2]$$

are not computably isometric.
These presentations add whole closed intervals each time α increases.

Can it be worse?

Theorem (Melnikov and N., 2012)

There are computable presentations L, R of a compact metric space so that no isomorphism is Δ^0_2.

The space is the closure of a computable sequence of elements $\sigma 0^\infty$ in Cantor space, for finite strings σ.

Can it be worse still?

No! We can bound the complexity of some isometry

Theorem

Suppose that a compact metric space has two presentations L, R. Then there is an isometry $g\colon L \to R$ with $g' \leq_T (L \oplus R)''$.

Proof. Since any self-embedding of a compact metric space is onto, it suffices to obtain an embedding $g\colon\ L \to R$ (then use symmetry).

Let L have dense sequence $(p_n)_{n\in\mathbb{N}}$. Start with an $L \oplus R$-computable double sequence y_i^n $(i < n)$ so that $y_0^n, \ldots, y_{n-1}^n$ realizes the same distances as $p_0, \ldots, p_{n-1}$, up to an error of 2^{-n}.

The embedding g is given as a path h on a finitely branching Π_1^0 tree relative to $(L \oplus R)'$, based on that double sequence. At each level we typically consider a finite open cover and need to pick a en element of it containing infinitely many elements of a computable sequence. We can choose g low in $(L \oplus R)'$ by the low basis theorem relative to $(L \oplus R)'$.

Some more detail on how to build embedding $g\colon L \to R$

- Let $A_\emptyset = \mathbb{N}$.
- A string $\sigma \in \omega^{<\omega}$ of length $2n$ can be extended by special point q of R such that

  $$d(y_n^k, q) < 2^{-n} \text{ for infinitely many } k \in A_\sigma.$$

  Enumerate these k into a set $C_{\sigma\hat{\ }q}$.
- A string $\eta \in \omega^{<\omega}$ of length $2n+1$ can be extended by a special point $\overline{r} = \langle r_0, \ldots, r_n \rangle$ of R^{n+1} such that

  $$d(\langle y_0^k, \ldots, y_n^k \rangle, \overline{r}) < 2^{-n-1} \text{ for infinitely many } k \in C_\eta.$$

  Enumerate these k into a set $A_{\eta\hat{\ }\overline{r}}$.

By compactness of all the R^m and the low basis theorem relativized, there is a path $h \in \omega^\omega$ of allowed extensions with $h' \leq_T (L \oplus R)''$.
From h we compute a double sequence $\langle w_i^n \rangle_{i \leq n, 0 < n}$ by letting

$$w_i^n = \text{the } i\text{-th component of } h(2n).$$

For each i: $(w_i^n)_{n>i}$ is a Cauchy name. So the point s $z_i := \lim_{n>i} w_i^n$ are uniformly computable in h. The map $g\colon L \to R$ given by $g(p_i) = z_i$ preserves distances.

Some directions and open questions

- Develop effective categoricity for compact computable metric groups and other metric structures.
- Study weaker senses of similarity for compact computable metric spaces: bi-Lipschitz equivalence, homeomorphism.
  In particular, is homeomorphism Σ^1_1-complete?

For the corresponding classical question, complexity of homeomorphism $\cong_h$ for compact metric spaces has not been determined. It is known that

$$E_0^+ \leq_B \cong_h \leq_B$$

isometry of metric spaces. The latter is easy because $X \cong_h Y$ iff $\mathcal{C}(X) \cong_i \mathcal{C}(Y)$. See Su Gao Ch 14 near end.