How the Lebesgue density theorem resolved the covering problem

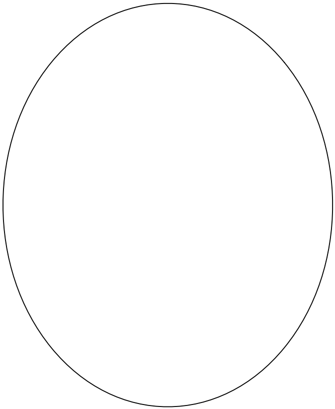
involves work with L. Bienvenu, N. Greenberg,
A. Kučera, D. Turetsky,
and work by A. Day and J. Miller

André Nies

ALC 2013, Guangzhou

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Basic objects of computability theory



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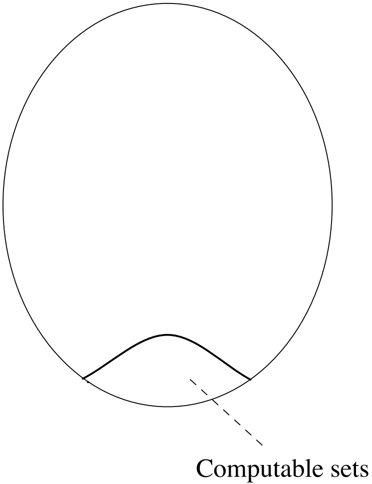

- The computable subsets of $\mathbb{N}$

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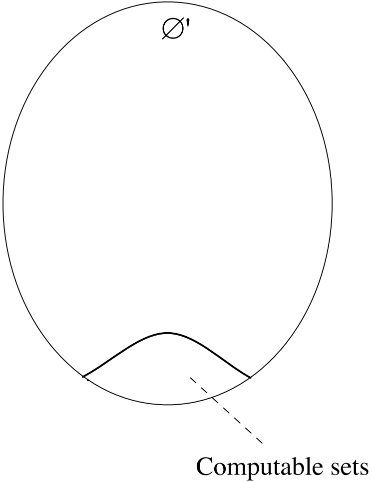

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- the halting problem $\emptyset'$

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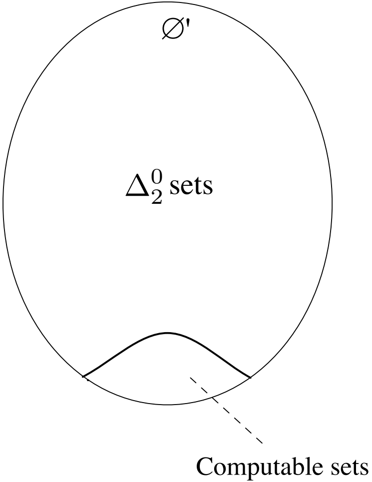

- The computable subsets of $\mathbb{N}$
- the halting problem $\emptyset'$
- Turing reducibility $\leq_T$
- the Δ^0_2 sets ($A \leq_{\mathrm{T}} \emptyset'$)

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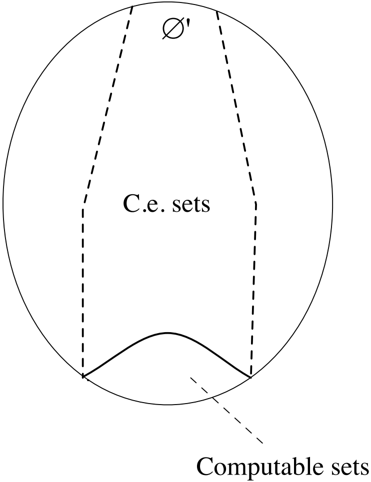

- The computable subsets of $\mathbb{N}$
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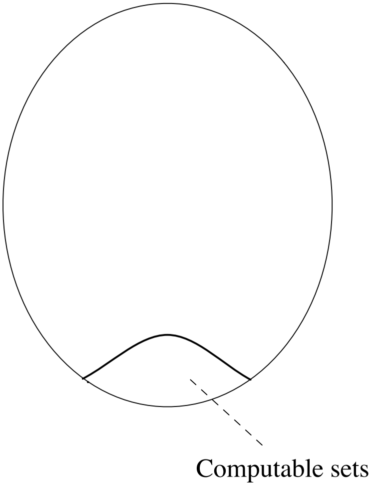
- Turing reducibility $\leq_T$
- the computably enumerable sets

Adding the world of (anti-)randomness



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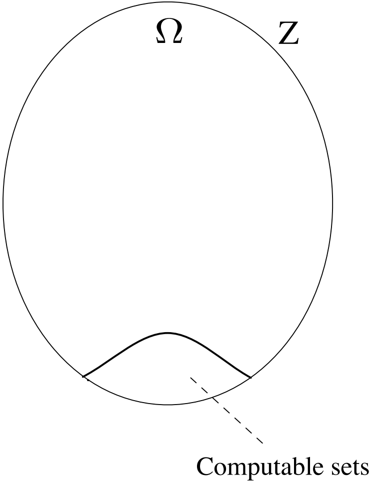

- The Martin-Löf random sets Z, such as Chaitin's halting probability Ω defined later on. We have $\Omega \equiv_T \emptyset'$.

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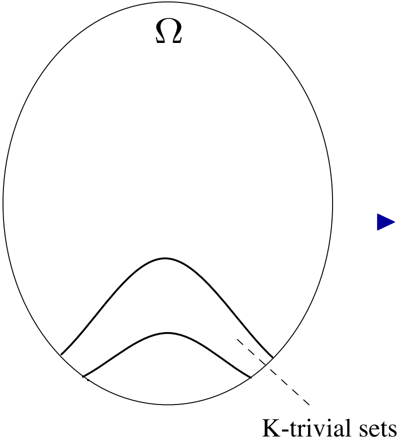

- The antirandom (K-trivial) sets.

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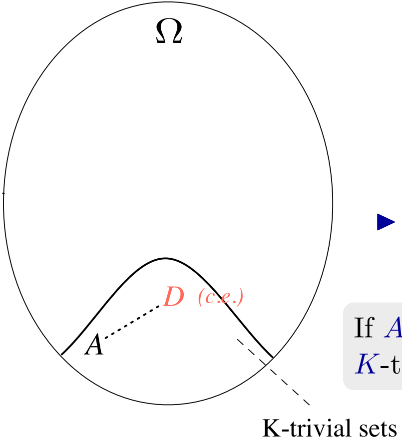
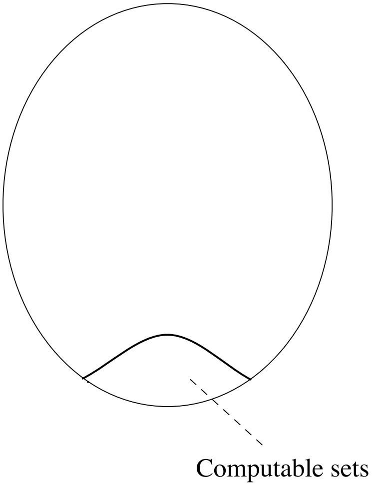


- The antirandom (K-trivial) sets.

If A is K-trivial, then there is c.e. K-trivial set $D \geq_{tt} A$. (Nies 2004)

Kučera's theorem and the covering problem



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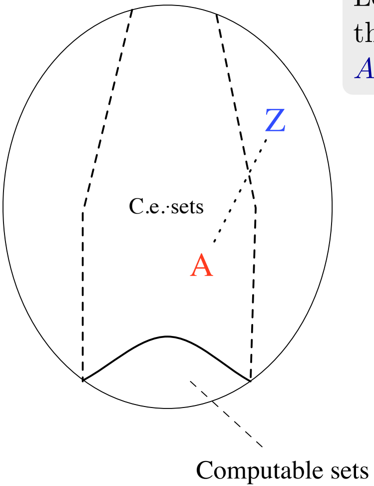

Let Z be a random Δ^0_2 set. Then there is a c.e., incomputable set $A \leq_{\mathrm{T}} Z$. (Kučera, 1986)

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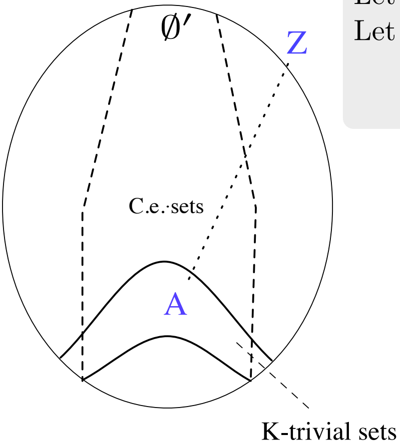

Let Z be random with $Z \not\geq_T \emptyset'$.
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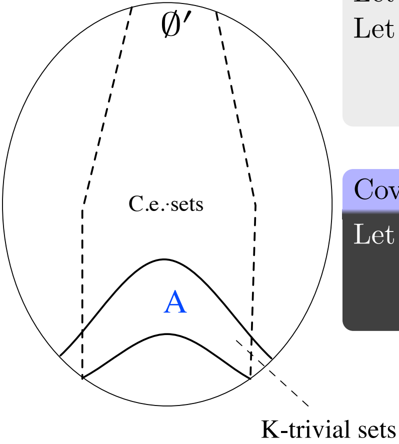

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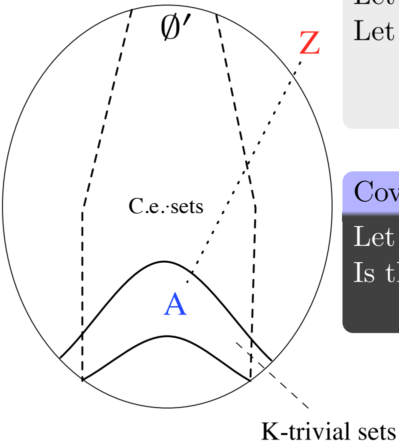

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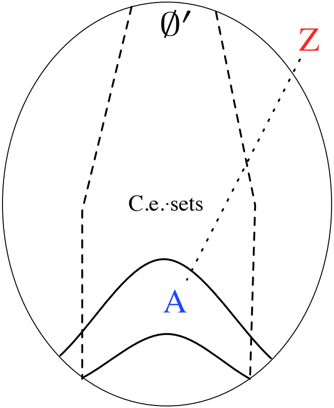

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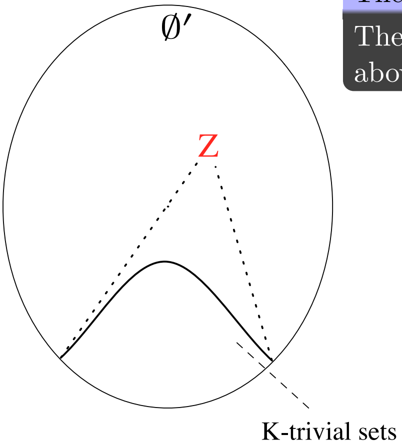
Let A be a c.e. K-trivial set.
Is there a ML-random $Z \geq_T A$ with $Z \not\geq_T \emptyset'$?

We may omit the assumption that A is c.e.: if not, replace A by a c.e. K-trivial set D above A.

A strong solution to the covering problem



Theorem — 5 + 2 authors

There is a ML-random set $Z <_T \emptyset'$ above all the K-trivials.

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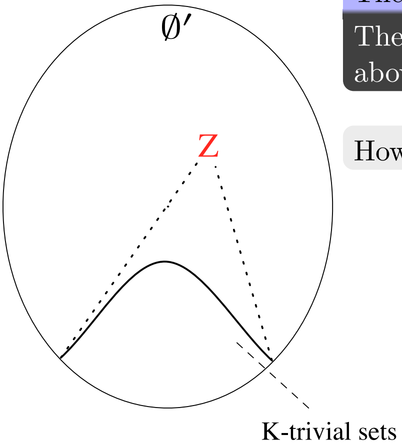

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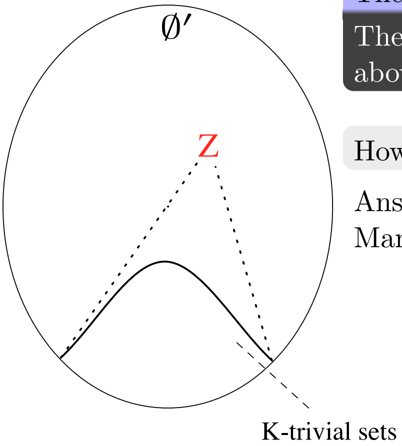

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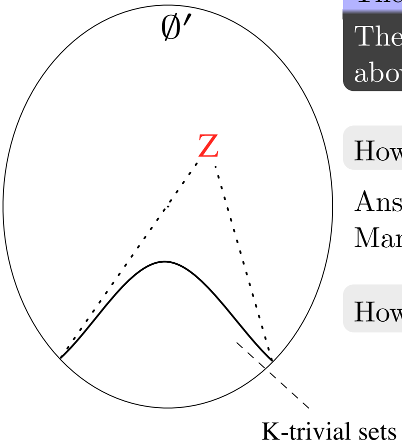

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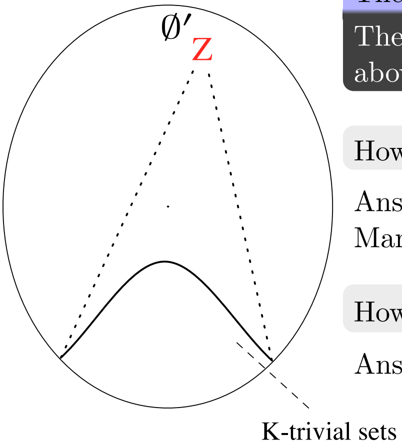

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Answer: not much more than Martin-Löf random.

How close to $\emptyset'$ must Z lie?

Answer: Z is very close to $\emptyset'$.

Background on random and antirandom sets

Martin-Löf's 1966 randomness notion

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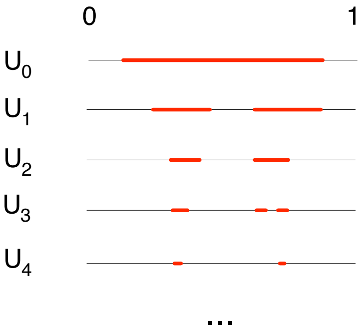

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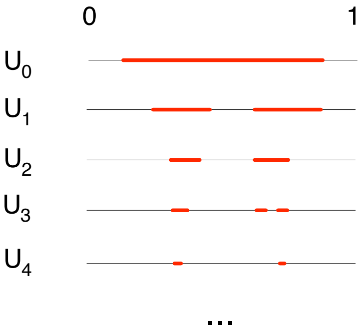

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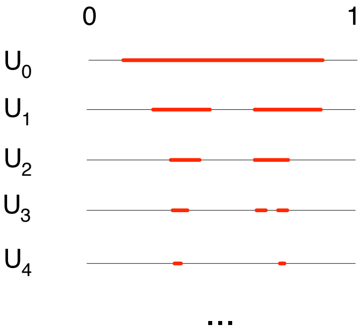
- Z passes the test if Z is not in all U_m.



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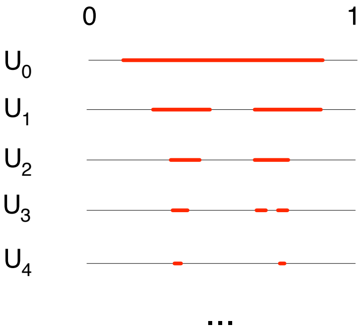
- Intuitively, U_m is an attempt to approximate a real z with accuracy 2^{-m}.
- Z passes the test if Z is not in all U_m.
- Z is called Martin-Löf random if it passes all ML-tests.



Descriptive string complexity K

Consider a partial computable function from binary strings to binary strings (called machine). It is called prefix-free if its domain is an antichain under the prefix relation of strings.

There is a universal prefix-free machine $\mathbb{U}$:
for every prefix-free machine M,

$$M(\sigma) = y \text{ implies } \mathbb{U}(\tau) = y \text{ for some } \tau \text{ with } |\tau| \leq |\sigma| + d_M,$$

and the constant d_M only depends on M.

The prefix-free Kolmogorov complexity of string y is the length of a shortest $\mathbb{U}$-description of y:

$$K(y) = \min\{|\sigma| : \mathbb{U}(\sigma) = y\}.$$

The Schnorr/Levin 1973 Theorem

We think of a string τ as random if it is incompressible:
$K(\tau) > |\tau| - b$ for some "small" constant b.
For an infinite sequence of bits Z, let

$$Z \upharpoonright_n = Z(0) \dots Z(n-1).$$

An infinite sequence of bits Z is Martin-Löf random iff each of its initial segments is random as a string:

Theorem (Schnorr 1973; Levin 1973)

Z is ML-random $\iff$ there is $b \in \mathbb{N}$ such that $\forall n\, [K(Z \upharpoonright_n) > n - b]$.

Chaitin's halting probability is ML-random:

$$\Omega = \sum \{2^{-|\sigma|} \colon \mathbb{U} \text{ halts on input } \sigma\}.$$

Definition of K-triviality

In the following, we identify a natural number n with its binary representation (as a string). For a string τ, up to additive const we have $K(|\tau|) \leq K(\tau)$, since we can compute $|\tau|$ from τ.

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An infinite sequence of bits A is K-trivial if, for some $b \in \mathbb{N}$,

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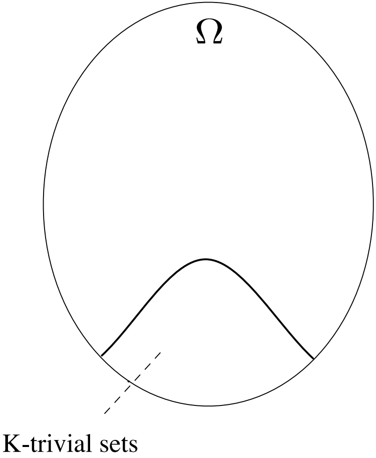
$$\begin{array}{lll} Z \text{ is random} & \Leftrightarrow & \forall n\, [K(Z\restriction_n) > n - O(1)] \\ A \text{ is } K\text{-trivial} & \Leftrightarrow & \forall n\, [K(A\restriction_n) \le K(n) + O(1)] \end{array}$$

Thus, being K-trivial means being *far from random*.

Some properties of the K-trivials



- Ideal in the Δ^0_2 Turing degrees (Chaitin '75, DHNS '03, N. '05)

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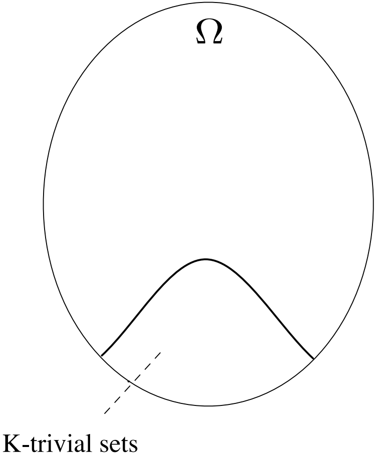

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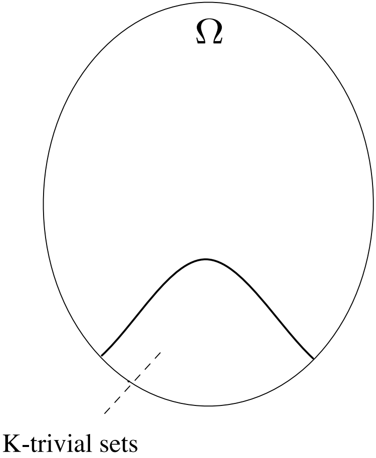

- Ideal in the Δ^0_2 Turing degrees (Chaitin '75, DHNS '03, N. '05)
- C.e. K-trivs have Σ^0_3 index set
- they are all superlow: $A' \leq_{tt} \emptyset'$ (N. '05)

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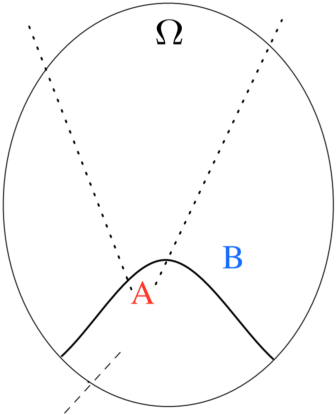

- Ideal in the Δ^0_2 Turing degrees (Chaitin '75, DHNS '03, N. '05)
- there is no largest: for every low c.e. set B, there is a K-trivial set $A \not\leq_T B$ (N., '02)

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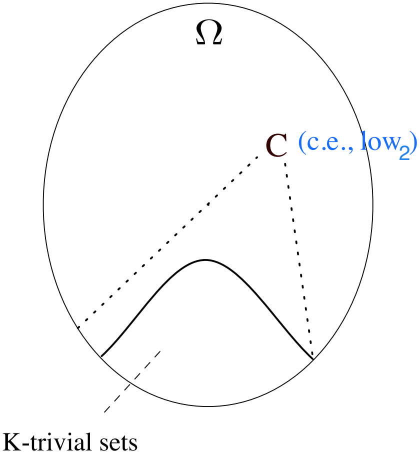

- Ideal in the Δ^0_2 Turing degrees (Chaitin '75, DHNS '03, N. '05)
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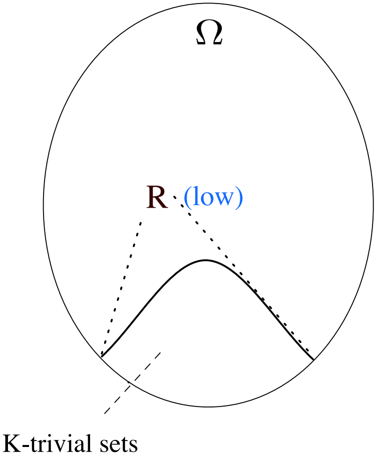

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- there is a low_1 set R above all of them (Kučera and Slaman)

Lebesgue Density

Density

Let λ denote uniform (Lebesgue) measure.

Definition

Let E be a subset of $[0,1]$. The *(lower) density* of E at a real z is

$$\rho(E \mid z) = \liminf_{z \in J\,,\, |J| \to 0} \frac{\lambda(J \cap E)}{|J|},$$

where J ranges over intervals.

This gauges how much of E is around z
as intervals zoom in on z.

It is the least fraction of E
in smaller and smaller intervals around z.

Lebesgue

From HENRI LEBESGUE, *Sur l' intégration des fonctions discontinues*, Annales scientifiques de l' É.N.S. 3e série, tome 27 (1910), p. 361-450; p. 407.

Raisonnant de même sur la densité à gauche, on voit finalement que *la densité d'un ensemble mesurable est égale à* un *en presque tous les points de cet ensemble.*

Translation:

Theorem (Lebesgue Density Theorem, 1910)

Let $E \subseteq [0,1]$ be measurable. For almost every $z \in [0,1]$: if $z \in E$, then E has density 1 at z.

Lebesgue's Theorem: towards an effective version

Recall: $\rho(E \mid z) = \liminf_{J \text{ interval}, z \in J, |J| \to 0} \lambda(J \cap E)/|J|$.

Theorem (Lebesgue Density Theorem, 1910)

Let $E \subseteq [0,1]$ be measurable. Then for almost every $z \in [0,1]$:
if $z \in E$, then $\rho(E \mid z) = 1$.

- If E is open, this is immediate, and actually holds for all $z \in [0,1]$.
- If E is closed, this is the simplest case where there is something to prove.

$E \subseteq [0,1]$ is effectively closed if one can list open intervals with union $[0,1] \setminus E$.

Definition

We say that a real z is a density-one point if $\rho(E \mid z) = 1$ for every effectively closed $E \ni z$.

Almost everywhere theorems and randomness

- Take an "almost everywhere" theorem from analysis, saying that a function, set, etc. is well-behaved at almost every real z.
- State an algorithmic version of the theorem.
- A sufficiently strong algorithmic randomness condition on z implies that the theorem holds at z.

Does Martin-Löf randomness ensure that an effectively closed $E \subseteq [0,1]$ has density one at $z \in E$?

Answer: NO!

Example

Let $E \neq \emptyset$, $E \subseteq [0,1]$ be an effectively closed set of Martin-Löf randoms. Let $z = \min(E)$. Then $\rho(E \mid z) = 0$

(This uses that every ML-random is Borel normal.)

Connecting density and K-triviality

This is based on the following work:

[Oberwolfach]	Bienvenu, Greenberg, Kučera, N. Turetsky	2012 (early)
[Berkeley]	Day and Miller	2012 (mid)
[Paris]	Bienvenu, Miller, Hölzl and N. (STACS 2012)	2011

Turing incompleteness and positive density

Definition

We say that a real z is a positive density point if
$\rho(E \mid z) > 0$ for every effectively closed $E \ni z$.

For a real $z \notin \mathbb{Q}$, let $Z \in 2^{\mathbb{N}}$ denote its binary expansion: $z = 0.Z$.

Theorem (Paris)

Let z be a Martin-Löf random real. Then
Z is NOT above the halting problem $\emptyset' \Leftrightarrow$
z is a positive density point.

This was applied to solve an open problem of Miller and N. (2006) on the interaction of K-trivials and randoms:
a K-trivial does not help a random to compute $\emptyset'$.

Theorem (Day and Miller, 2012)

Let $A \subseteq \mathbb{N}$ be K-trivial. Suppose $Z \subseteq \mathbb{N}$ is a Martin-Löf random set such that $Z \oplus A \geq_T \emptyset'$. Then already $Z \geq_T \emptyset'$.

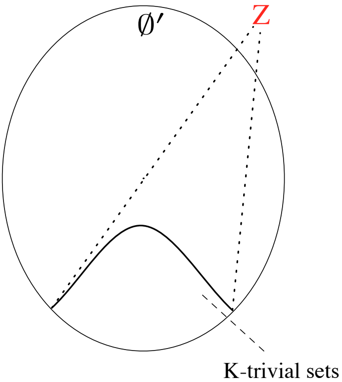
The main connection of density and K-trivials

Recall: $\rho(E \mid z) = \liminf_{|J|\to 0, z\in J} \lambda(J \cap E)/|J|$.

Definition (Recall)

We say that a real z is a density-one point if
$$\rho(E \mid z) = 1 \text{ for every effectively closed } E \ni z.$$

In other words, z satisfies the Lebesgue Theorem for effectively closed sets.



Theorem (Oberwolfach)

Let z be a Martin-Löf random real. Suppose z is NOT a density-one point.

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It is easier to work in Cantor space $2^{\mathbb{N}}$. For a string σ, let

$$[\sigma] = \{X \in 2^{\mathbb{N}} \colon\ X \succ \sigma\}$$

For $E \subseteq 2^{\mathbb{N}}$ and $Z \in 2^{\mathbb{N}}$, let the lower dyadic density be

$$\rho_2(E \mid Z) = \liminf_{\sigma \prec Z\,,\,|\sigma| \to \infty} \lambda([\sigma] \cap E) 2^{-|\sigma|};$$

Khan and Miller have shown that for ML-random Z, dyadic density-one points are already full density-one points. So the transition to Cantor space is fine.

We use a new notion called Oberwolfach (OW) randomness. We have

$$\text{OW-random} \Rightarrow \text{ML-random},$$

but not conversely.

Oberwolfach randomness

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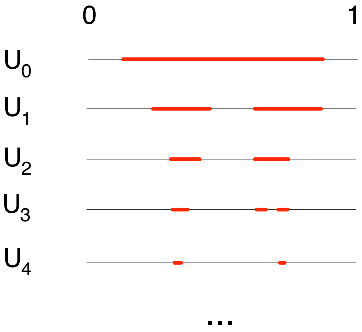

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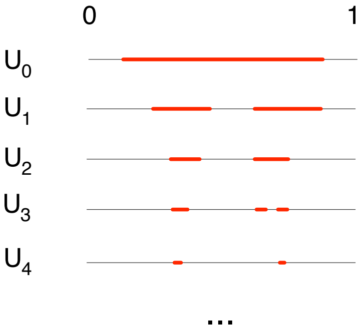
- Z passes the test if Z is not in all U_n.



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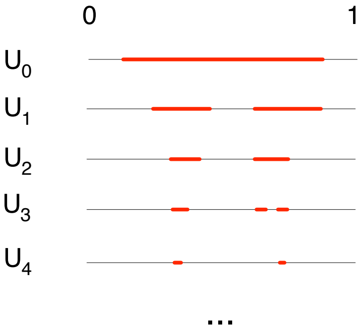
- Z passes the test if Z is not in all U_n.
- Z is Oberwolfach random if it passes all left-c.e. bounded tests.



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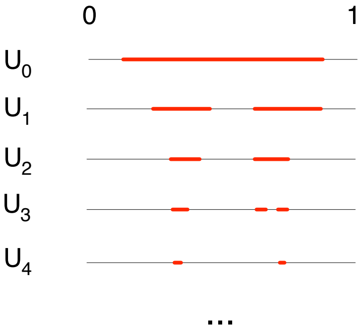


We can use a similar definition in Cantor space $2^{\mathbb{N}}$. They are equivalent via the binary expansion.

There are two parts to the argument to establish that every ML-random non-density-one point Z is above all the K-trivials.

Part 1: If $E \subseteq 2^{\mathbb{N}}$ is a Π^0_1 class, $Z \in E$, and $\rho_2(E \mid Z) < 1$,

then Z is not OW random.

Part 2: If Z is ML-random, but not OW random,

then Z is above all the K-trivials.

Part 1. If $E \subseteq 2^{\mathbb{N}}$ is a Π^0_1 class, $Z \in E$ and $\rho_2(E \mid Z) < \Theta < 1$ then Z is not OW random.

The Oberwolfach group used convergence of left-c.e. martingales.

J. Miller's later proof:

For $S \subseteq 2^{<\omega}$ let $[S]^{\prec}$ denote the open set it generates in Cantor space. Let $\lambda_\sigma(E)$ be the local measure $2^{-|\sigma|}\lambda(E \cap [\sigma])$.

Let $\sigma_0, \sigma_1, \ldots$ be a prefix-free sequence of strings such that $2^{\mathbb{N}} \setminus E = \bigcup_n [\sigma_n] = [\sigma_0, \sigma_1, \ldots]^{\prec}$.

Let $\beta_n = \lambda[\sigma_0, \ldots, \sigma_{n-1}]^{\prec}$, and $\beta = \sup_n \beta_n$

$$U_n = \{Y \colon \exists k \, \lambda_{[Y\restriction_k]}([\sigma_n, \sigma_{n+1}, \ldots]^{\prec}) > 1 - \Theta\}.$$

- Since $Z \in E$, and $(\sigma_i)_{i \in \mathbb{N}}$ is prefix free, $Z \in U_n$ for each n
- $\lambda(U_n) \leq \frac{1}{1-\Theta}(\beta - \beta_n)$.

So Z is not OW random.

Part 2: If Z is ML-random, but not OW random, then Z is above each K-trivial set A.

Suppose $\beta = \sup \beta_n$ for an increasing computable sequence $(\beta_n)_{n\in\mathbb{N}}$ of rationals. Suppose $Z \in \bigcap U_n$ where $\lambda(U_n) \le \beta - \beta_n$.

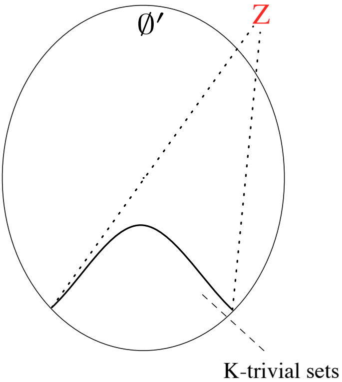
Since A is K-trivial, by a result of N. (2010), there is a computable enumeration (A_s) of A such that

$$\infty > \sum_s \{\beta_s - \beta_n \colon \ n \text{ is least such that } A_s(n) \neq A_{s-1}(n)\}.$$

We define a Turing functional, and declare that all the oracles in $U_{n,s-1}$ compute $A_{s-1}(n)$. If $A_s(n) \neq A_{s-1}(n)$, let $\mathcal{V}_s = U_{n,s-1}$ be the class of oracles that are now incorrect.

Since $\lambda U_{n,s-1} \le \beta_s - \beta_n$, we have $\sum_s \lambda \mathcal{V}_s < \infty$. So $(\mathcal{V}_s)_{s\in\mathbb{N}}$ is a "Solovay test": If the oracle Z is incorrect infinitely often, it fails this test, so it is not ML-random. Thus $A \le_{\mathrm{T}} Z$.

The main connection of density and K-trivials



We have proved:

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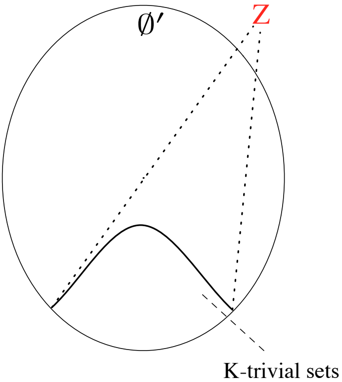

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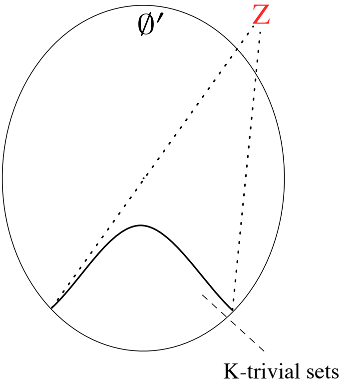
[Paris] in their journal version subsequently give a direct proof, not using OW-randomness.
Rather than K-trivial, it is based on the equivalent lowness for MLR.
The proof given here also leads to a new proof that K-trivial implies low for MLR.

The main connection of density and K-trivials



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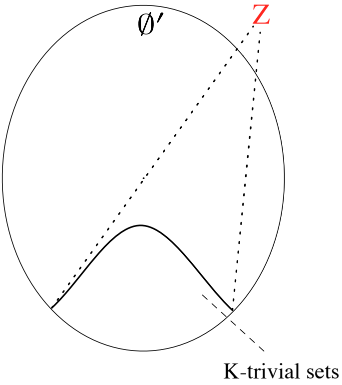
Does Z as in the picture exist?

The main connection of density and K-trivials



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Where do we get a ML-random set $Z \not\geq_T \emptyset'$ that is not a density-one point?

Why such a Z exists

Theorem (Berkeley, i.e., Day and Miller)

Let P be a nonempty Π^0_1 class of ML-randoms. There is a ML-random set $Z \not\geq_T \emptyset'$ such that $\rho_2(P \mid Z) \leq 1/2$.

[Paris] characterized difference randomness of a ML-random Z via positive density:

$$Z \not\geq_T \emptyset' \text{ iff } Z \text{ is a positive density point.}$$

Berkeley built a set Z that is a positive density point.

Note that, for the Day-Miller set Z, the local measure $\lambda_\sigma(Z)$ for $\sigma \prec Z$ oscillates between 1 (asymptotically), and a value ϵ with $0 < \epsilon \leq 1/2$.

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Proof. Force with conditions of the form $\langle \sigma, Q \rangle$, where

- σ is a string, $Q \subseteq P$, $[\sigma] \cap Q \neq \emptyset$
- there is $\delta < 1/2$ such that each string $\tau \succeq \sigma$ has two options:

$$\text{either } [\tau] \cap Q = \emptyset, \text{ or } \lambda_\tau(Q) \geq \lambda_\tau(P) - \delta.$$

(Q either loses all, or $\leq \delta$ of P's local measure within $[\tau]$.)

$\langle \sigma', Q' \rangle$ extends $\langle \sigma, Q \rangle$ if $\sigma' \succeq \sigma$ and $Q' \subseteq Q$.
We have an initial condition $\langle \emptyset, P \rangle$ (via $\delta = 0$).

If G is a sufficiently generic filter, then $Z_G = \bigcup \{\sigma \colon \langle \sigma, Q \rangle \in G\}$ is a ML-random positive density point, and $\rho_2(P \mid Z) \leq 1/2$.
Then by Bienvenu et al., $Z \not\geq_T \emptyset'$.

The strongest answer to the covering question

Berkeley's careful effectvization of the forcing yields a Δ^0_2 set Z.

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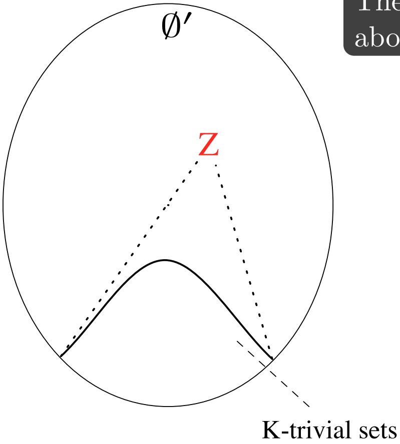

Theorem (Oberwolfach + Berkeley)

There is a ML-random set $Z <_T \emptyset'$ above all the K-trivials.

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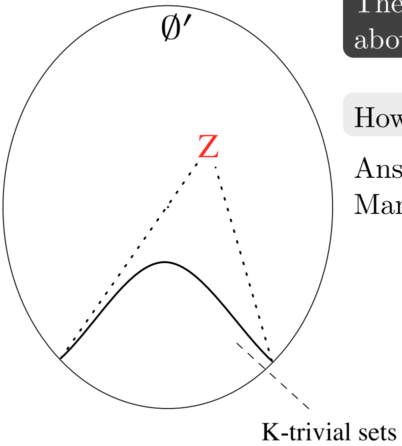

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There is a ML-random set $Z <_T \emptyset'$ above all the K-trivials.

How random can Z be?

Answer: not much more than Martin-Löf random.

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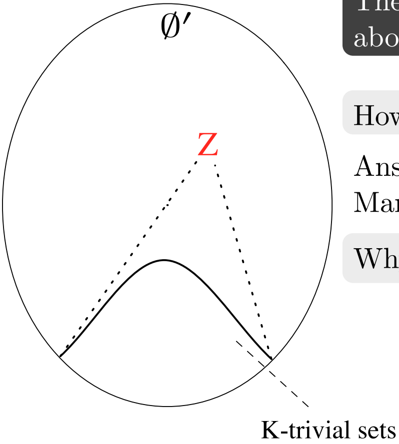

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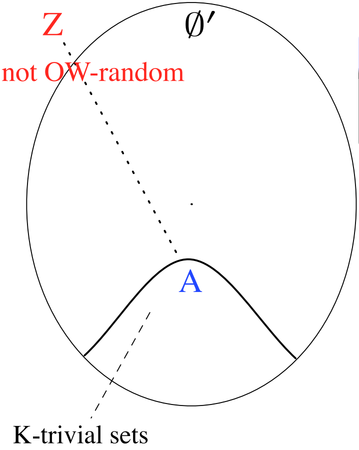
Answer: not much more than Martin-Löf random.

Why?

Z cannot be OW-random



Theorem (Oberwolfach)

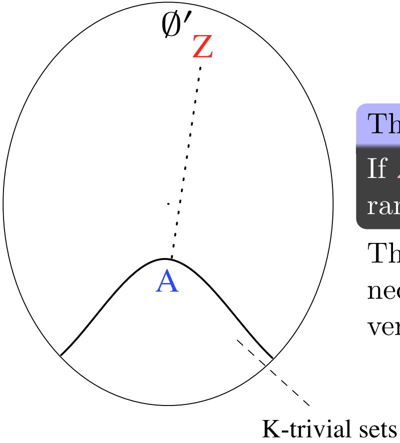
There is a K-trivial set A so that no set $Z \geq_T A$ is OW-random.

So Z is ML-random and incomplete (difference random), but not much more.

We call A a "smart" K-trivial: any random set above A is above all the K-trivials.

Z must be very close to $\emptyset'$

Recall that Y is LR-hard if every Y-random set is random relative to $\emptyset'$. Such a set is superhigh: $\emptyset'' \leq_{tt} Y'$. A ML-random LR-hard set $Y \not\geq_T \emptyset'$ was built by N. (2009); Kučera.



Theorem (Oberwolfach)

If Z is ML-random but not OW-random, then Z is LR-hard.

Thus, the set $Z \geq_T A$ is necessarily LR-hard, and so, very close to $\emptyset'$.

Density randomness

Definition

We say that Z is density random if Z is ML-random and $\rho_2(P \mid Z) = 1$ for each Π^0_1 class $P \in Z$.

We can equivalently require density 1 in the setting of reals.

Theorem

The following are equivalent for $Z \in 2^{\mathbb{N}}$, $z = 0.Z$.

- Z is density random
- [Madison group, 2012] Each left-c.e. martingale M converges: $\lim_n M(Z \upharpoonright_n)$ exists ($M(\sigma)$ is a c.e. real uniformly in string σ)
- [N., 2013] $f'(z)$ exists for each interval-c.e. function f (basically, the variation function of a computable function)
- [Miyabe, 2013] z is a Lebesgue point for each integrable lower semicomputable function $f\colon [0,1] \to \overline{\mathbb{R}}^+$.

Randomness in the higher setting

- N. and Hjorth (2007) studied randomness using tools from effective descriptive set theory, rather than algorithmic tools. The idea is to replace c.e. objects by Π^1_1 objects in the definitions of tests.
- There is much recent work on this: Chong and Yu; Bienvenu, Greenberg and Monin.

Theorem (Greenberg and N., 2013, based on Madison)

The following are equivalent for $Z \in 2^{\mathbb{N}}$, $z = 0.Z$.

- Z is Π^1_1-ML random and $\rho_2(\mathcal{C} \mid Z) = 1$ for each closed Σ^1_1 class $\mathcal{C} \ni Z$.
- The same without "closed"
- Every left-Π^1_1 martingale converges along Z.

Questions on separating the new randomness notions

Question (Oberwolfach, Miller)

Is the implication

$$\text{Oberwolfach random} \Rightarrow \text{density random}$$

a proper one? Equivalently, can a ML-random Z above all the K-trivials be density random?

Question (Oberwolfach)

Is the implication

$$\text{ML-random} \wedge \text{not LR-hard} \Rightarrow \text{Oberwolfach random}$$

a proper one?

Questions on density randomness

Question (Turetsky)

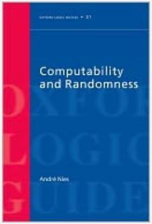
Is density randomness closed downward within the ML-randoms?

This is known for most randomness notions stronger than Martin-Löf's, including for OW-randomness (by the results above).

Question (Franklin)

Is density randomness equivalent to being a Birkhoff point for each computable measure preserving operator and semicomputable function?

Book references for background



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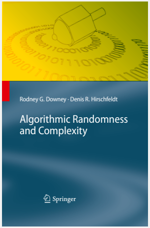

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