Complexity of isomorphism relations

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Abstract: Given a class of structures encoded by reals, how can we determine the complexity of the corresponding isomorphism relation? For instance, isometry of Polish spaces is Borel complete for orbit equivalence relations (Gao and Kechris, 2003), while isomorphism of countable graphs is not (H. Friedman and Stanley, 1989).
We mainly consider a modified form of the question posed above, where definability, or effectiveness constraints are placed on the class, the isomorphism, or both. For instance, we study isometry of computable compact metric spaces. Related to this we discuss the Scott analysis of Polish spaces.
We show that computable isomorphism of computable Boolean algebras is a Sigma-0-3 complete equivalence relation on N, where the reductions are computable functions. We give further examples of complete equivalence relations at lower levels; in particular, there is one at level Pi-0-1.
This project involves lots of co-workers, including Sy Friedman, K. Fokina, M. Koerwien, A. Melnikov, R. Miller, Selwyn Ng, P. Schlicht,

Motivating question

Question

Suppose we are given a class $\mathcal{K}$ of mathematical structures.

How hard is it to determine whether two structures in $\mathcal{K}$ are isomorphic?

- I will first consider this in the classic setting.
- Thereafter, I will discuss the case that the structures, or the isomorphisms, or both, are in some sense effectively presented.

Borel reducibility

The complexity of an equivalence relation is often determined by being a hardest object in a natural class. E.g. universal countable Borel equivalence relation.

We compare the complexity of equivalence relations using Borel reducibility.

Definition (H. Friedman, Stanley, 1989)

Let $\mathcal{X}, \mathcal{Y}$ be Polish spaces.

Let E be an equivalence relation on $\mathcal{X}$.

Let F be an equivalence relation on $\mathcal{Y}$.

We write $E \leq_B F$ if there is a Borel map $\phi \colon \mathcal{X} \to \mathcal{Y}$ such that

$$u \, E \, v \iff \phi(u) \, F \, \phi(v).$$

Orbit equivalence relations

Definition

An equivalence relation E on a Polish Space $\mathcal{X}$ is called orbit equivalence relation if for some Polish group G acting continuously on $\mathcal{X}$, we have

$$xEy \iff \exists g \in G\,[g \cdot x = y].$$

- E is analytical, but often not Borel
- every equivalence class $[x]_E$ is Borel.
  Proof: Let G_x be the stabilizer of x, which is closed.
  Lusin-Souslin: the range of a 1-1 Borel map is Borel.
  Apply this to the natural map $G/G_x \to [x]_E$.
  (This actually works for Borel actions.)

Three isomorphism relations

of different complexity

1. Graph isomorphism

We consider graphs with domain ω. Isomorphism of graphs is an orbit equivalence relation via the action of S_∞, the group of permutations of ω.

Let $\mathbb{E}_1$ denote almost equality of elements of $\mathbb{R}^\omega$ (i.e., sequences of reals). Friedman and Stanley (1989) showed that

$$\mathbb{E}_1 \not\leq_B \text{graph isomorphism}.$$

In fact, $\mathbb{E}_1$ is not reducible to any orbit equivalence relation.

Let c_0 denote the Borel orbit equivalence relation on $\overline{x}, \overline{y} \in \mathbb{R}^\omega$ that $\lim_n (x_n - y_n) = 0$. By the Hjorth turbulence theorem, $c_0 \not\leq_B$ any orbit equivalence relation given by an S_∞ action. In particular,

$$c_0 \not\leq_B \text{graph isomorphism}.$$

2. Isometry of Polish metric spaces

A Polish metric space $\mathcal{X} = (X, d)$ with dense sequence $(p_n)_{n\in\mathbb{N}}$ can be encoded by a single element of $\mathbb{R}^{\omega\times\omega}$, namely $d(p_i, p_k)_{i,k\in\omega}$.

Theorem (Gao-Kechris 2003/Clemens)

Let E be any orbit equivalence relation. Then

$$E \ \leq_B \cong_i$$

where $\cong_i$ denotes isometry of Polish metric spaces.

- In particular, $c_0 \ \leq_B \cong_i$. Thus, isometry of Polish spaces is strictly more complex than graph isomorphism.
- They also showed that $\cong_i \leq_B$ the orbit equivalence relation given by the $\mathrm{Iso}(\mathbb{U})$ action on $F(\mathbb{U})$, the Effros structure of the Urysohn space.

3. Isometry of compact metric spaces

For a compact metric space $\mathcal{X}$, let $D_n(\mathcal{X})$ denote the set of all $n \times n$ matrices $d(x_i, x_k)_{i,k<n}$, where $x_0, \ldots, x_{n-1} \in \mathcal{X}$.

Theorem (Gromov, 1999)

Let $\mathcal{X}_0, \mathcal{X}_1$ be compact metric spaces such that $D_n(\mathcal{X}_0) = D_n(\mathcal{X}_1)$ for each n. Then $\mathcal{X}_0$ and $\mathcal{X}_1$ are isometric.

- The sequence of compact sets $D_n(\mathcal{X}) \subseteq \mathbb{R}^n$ can be encoded by a single point in a standard Polish space.
- So, $\cong_i$ on compact spaces is smooth, that is, Borel reducible to the identity on $\mathbb{R}$.
- This means it is much simpler than graph isomorphism.

Scott analysis for Polish metric spaces

α-equivalence of tuples in structures

Definition

Let M, N be $\mathcal{L}$-structures. Let $\bar{a}, \bar{b}$ tuples of the same length from M, N.

- $\bar{a} \equiv_0 \bar{b}$ if the quantifier-free types of the tuples are the same.
- For a limit ordinal α, $\bar{a} \equiv_\alpha \bar{b}$ if $\bar{a} \equiv_\beta \bar{b}$ for all $\beta < \alpha$.
- $\bar{a} \equiv_{\alpha+1} \bar{b}$ if both of the following hold:
  - For all $x \in M$, there is some $y \in N$ such that $\bar{a}\, x \equiv_\alpha \bar{b}\, y$
  - For all $y \in N$, there is some $x \in M$ such that $\bar{a}\, x \equiv_\alpha \bar{b}\, y$

Back-and-forth systems and Scott rank

- A back-and-forth system for a pair of structures M, N is a set of finite partial isomorphisms with the two-sided extension property.
- Recall that $M \cong_p N$ means that there is a nonempty back-and-forth system for the two structures.
- Suppose α is least such that $\equiv_\alpha$ implies $\equiv_{\alpha+1}$ for all tuples in M, N. If $\equiv_\alpha$ contains $\langle \emptyset, \emptyset \rangle$ then we get a non-empty back-and forth system, so $M \cong_p N$.
- For $M = N$, α is called the Scott rank $\mathrm{sr}(M)$.
  Note that $\mathrm{sr}(M) < |M|^+$.

Metric spaces as structures in first-order language

We view a metric space (X, d) as a structure for the signature

$$\{R_{<q}, R_{>q} \colon\ q \in \mathbb{Q}^+\},$$

where $R_{<q}$ and $R_{>q}$ are binary relation symbols.

- The intended meaning of $R_{<q}xy$ is that $d(x, y) < q$.
- The intended meaning of $R_{>q}xy$ is that $d(x, y) > q$.

Clearly, isomorphism is isometry.

Polish spaces vs. countable structures

Sometimes, Polish metric spaces behave like countable structures.
For instance, for countable structures A, B we have

$$A \cong_p B \Rightarrow A \cong B.$$

This can be easily extended to Polish metric spaces. In particular, we can view the Scott relations $\equiv_\alpha$ as approximations to isometry.

Sometimes, they don't. We have already seen that

$$\text{graph isomorphism} <_B \text{isometry of Polish spaces.}$$

The left side is a universal S_∞ orbit equivalence relation. The right side Borel reduces all orbit equivalence relations.

Polish metric spaces of low Scott rank

The Urysohn space has Scott rank 0: if tuples $\bar{a}, \bar{b}$ realize the same distances, then they are isometric.

Theorem (extending Gromov's argument)

A compact metric space M has Scott rank at most ω.

Proof:

- $\bar{a} \equiv_\omega \bar{b}$ implies that for each tuple $\widetilde{x}$ there is a tuple $\widetilde{y}$ of the same length such that $\bar{a}\,\widetilde{x}$ realizes the same distances as $\bar{b}\,\widetilde{y}$; and conversely.
- By Gromov's argument, this implies there is an isometry of M sending $\bar{a}$ to $\bar{b}$.

By the same argument, a compact metric space is determined by its existential positive theory. Thus it has a rather simple Scott sentence within the Polish spaces.

Discrete ultrametric spaces can have arbitrary countable Scott rank

Theorem (S. Friedman, M.Koerwien, N.)

For each $\alpha < \omega_1$, there is a discrete ultrametric space M of Scott rank $\alpha \cdot \omega$.

M is given as the maximal paths on a subtree of $\omega^{<\omega}$. For $\sigma \neq \tau \in M$, the distance is 2^{-k} where k is the least disagreement.

Upper bounds on the Scott rank

- By cardinality considerations, the Scott rank of a Polish space is $< (2^\omega)^+$.
- In fact, the Scott rank is less than the least ordinal α such that $L_\alpha(\mathbb{R}) \models$ Kripke-Platek set theory (i.e., with Σ_1 replacement and Δ_0 comprehension).

Question

Is the Scott rank of every Polish metric space countable?

- By Gao/Kechris (2003) the isometry class of each Polish space is Borel. This suggests an affirmative answer by analogy with the case of countable structures.
- There are possible connections to the topological Scott analysis of Hjorth for general Polish group actions.

In the second part of the talk, the structures, or the isomorphisms, or both, are in some sense effectively presented.

Effectivizing structures, or isomorphisms

- Camerlo (2002) studied computable isomorphism between arbitrary structures with domain contained in ω. He showed that for many classes, such as groups, fields, linear orders, Boolean algebras, this equivalence relation is universal countable. He reduced recursive isomorphism on 5^ω to it.
- In contrast, we will consider computable structures with arbitrary, and then with effective isomorphisms.
- This is related to a program to understand the complexity of equivalence relations on ω using computable reductions, analogous to the case of Borel reductions.
- Sy Friedman, E. Fokina, J. Hamkins, R. Miller, A. Sorbi, A. Törnquist and others have worked in this.

Definition of computable structure

The atomic diagram of a structure M is the set of all basic facts that hold in M: $R^M a_1, \ldots a_n$, $f^M(a,b) = c$ etc.

Definition

Let $\mathcal{L}$ be a computable language. An $\mathcal{L}$-structure with domain $\subseteq \omega$ is called **computable** if its atomic diagram is computable.

Examples: ring of integers, countable free groups, many Boolean algebras, ...

A computable structure is given by a computable index, that is a Turing machine deciding the atomic diagram.

Arbitrary isomorphisms (1)

- Let $I(\mathcal{K})$ be the set of indices of computable structures in a class $\mathcal{K}$. We will assume $I(\mathcal{K})$ is hyperarithmetical (usually it is in fact arithmetical).
- Note that isomorphism on $I(\mathcal{K})$ is a Σ^1_1 relation on numbers.
- Fokina, S. Friedman, Harizanov, Knight, McCoy, Montalban 2010 studied reductions between equivalence relations that are given by a partial computable function with domain containing the relevant set $I(\mathcal{K})$. (This reducibility is denoted $\leq_{FF}$.)

Theorem (Fokina, S. Friedman et al., 2010)

Each Σ^1_1 equivalence relation is reducible in the sense of $\leq_{FF}$ to
(a) isomorphism of computable graphs
(b) isomorphism of computable subtrees of $\omega^{<\omega}$.

Arbitrary isomorphisms (2)

Fokina, Friedman et al. posed a "Post problem" for isomorphism relations:

Question

Let $\mathcal{K}$ be a reasonable class such that isomorphism on $I(\mathcal{K})$ is not hyperarithmetical. Is isomorphism already Σ^1_1 complete for $\leq_{FF}$?

A potential counterexample would be the class $\mathcal{K}$ of Boolean algebras. Note that $I(\mathcal{K})$ is Π^0_2.

Isomorphism of computable Boolean algebras is known to be

- m-complete for Σ^1_1 sets (Goncharov and Knight, 2002), but
- not known to be complete for Σ^1_1 equivalence relations.

The arithmetical hierarchy

We study whether computable structures are effective isomorphic. The right setting to measure complexity of effective isomorphism is the arithmetical hierarchy.

$$\begin{array}{ccccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \Sigma^0_3 & & \\
 & \subset & & \subset & & \subset & & \subset & & \subset & & \subset & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & & & \cdots \\
 & \subset & & \subset & & \subset & & \subset & & \subset & & \subset & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & \Pi^0_3 & &
\end{array}$$

For instance, to say e is an index of a computable structure is Π^0_2. The relation $e \cong_{comp} i$ that indices e, i describe computably isomorphic structures is Σ^0_3.

For equivalence relations E, F on ω, we write $E \leq_c F$ if there is a computable function ϕ such that $u\,E\,v \iff \phi(u)\,F\,\phi(v)$.

Computable isomorphism

Theorem (S. Friedman, Fokina, N)

$\equiv_1$ on r.e. sets is complete for Σ^0_3 equivalence relations.

Corollary

Computable isomorphism on computable equivalence relations with all classes of size at most 2 is Σ^0_3 complete.

Proof of Corollary.

Given r.e. set A, build a computable equivalence relation R_A such that

$$A \equiv_1 B \text{ iff } R_A \equiv_{comp} R_B.$$

We may assume at most one element enters A at each stage, and only at even stages.

Let $R_A = \{\langle 2a, 2t+1\rangle \colon a \text{ enters } A \text{ at stage } t\}$.

Boolean algebras and Σ^0_3 completeness

Theorem (Fokina, Friedman, N)

Computable isomorphism of computable Boolean algebras is complete for Σ^0_3 equivalence relations.

Proof. Reduce 1-equivalence $\equiv_1$ of r.e. sets W^e containing the evens. Define the Boolean algebra B^e to be the interval algebra of a computable linear order $\bigoplus_{x \in \omega} M^e_x$, where

- M^e_x has one element, until x enters W^e;
- when that happens, expand M^e_x to a computable copy of $[0, 1)_{\mathbb{Q}}$.

One can show that $W^e \equiv_1 W^i \iff B^e \cong_{comp} B^i$.

Computable metric spaces

- A Polish metric space $\mathcal{X}$ with dense sequence $(p_n)_{n\in\mathbb{N}}$ is computable if the atomic diagram for this sequence (consisting of facts such as $d(p_i, p_j) > 1/3$) is computable. Call the atomic diagram for a particular sequence a presentation of the space.
- Let $\mathcal{Y}$ have dense sequence $(q_k)_{k\in\mathbb{N}}$. A Lipschitz map $\phi\colon \mathcal{X} \to \mathcal{Y}$ is computable if there is a computable function α such that

$$\phi(p_i) = \lim_n q_{\alpha(i,n)} = y,$$

  where $d(y, q_{\alpha(i,n)}) \leq 2^{-n}$. That is, from input i we can compute a "Cauchy name" for $\phi(p_i)$.

Examples: $\mathbb{R}$, $\mathcal{C}[0,1]$, and other natural Polish metric spaces.

Proposition

Computable isometry of computable metric spaces is Σ^0_3 complete.

Variant of the motivating question

Question

Suppose we are given a class $\mathcal{K}$
of mathematical structures.

Given presentations of isomorphic structures in $\mathcal{K}$,
can we determine an isomorphism from these presentations?

- If so, then it should also be comparatively easy to determine whether two structures in $\mathcal{K}$ are isomorphic.
- We will see that this is the case for compact metric spaces.

Computable categoricity

An abstract Polish metric space is called computably categorical (c.c.) if any two of its computable presentations are computably isomorphic. A. Melnikov studied this in his thesis (supervised by Khoussainov and me).

Theorem (A. Melnikov)

The following computable metric spaces are c.c.:

- *The unit interval; Cantor space;* ℓ_2
- *Urysohn space.*

Theorem (A. Melnikov)

The following computable metric spaces are not c.c.:

- ℓ_1 (Pour-El and Richards, essentially)
- $\mathcal{C}[0,1]$

Compact metric spaces

Isometry of compact metric spaces is very simple from the classical point of view:

- It is smooth.
- The Scott rank is at most ω.
- Isometry is a closed equivalence relation on $K(\mathbb{U})$, the space of compact subsets of Urysohn space.

Recursion theory provides a different view.

Theorem (Melnikov and N., 2012)

There is a compact metric space with two computable presentations L, R so that no isomorphism is Δ^0_2.

The space is in fact quite simple: it is the closure of a computable sequence of rationals in $[0, 1]$.

Isomorphism of low complexity

Theorem (Melnikov and N., 2012)

Suppose that a compact metric space has two presentations L, R.

- *Then there is an isometry $g \colon L \to R$ with $g' \leq_T (L \oplus R)''$. That is, g is low relative to $(L \oplus R)'$.*
- *In particular, if L, R are computable then g is Δ^0_3.*

Isomorphism of low complexity

Theorem (Melnikov and N., 2012)

Suppose that a compact metric space has two presentations L, R. Then there is an isometry $g\colon L \to R$ with $g' \le_T (L \oplus R)''$.

Proof. Since any self-embedding of a compact metric space is onto, it suffices to obtain an embedding $g\colon L \to R$ (then use symmetry).

Let L have dense sequence $(p_n)_{n \in \mathbb{N}}$. Start with an $L \oplus R$-computable double sequence y_i^n $(i < n)$ so that $y_0^n, \ldots, y_{n-1}^n$ realizes the same distances as $p_0, \ldots, p_{n-1}$, up to an error of 2^{-n}.

g is given as a path on a finitely branching Π_1^0 tree relative to $(L \oplus R)'$, based on that double sequence. At each level we typically consider a finite open cover and need to pick a piece of it containing infinitely many elements of a sequence. We can chose g low in $(L \oplus R)'$ by the low basis theorem.

Descriptive complexity of the isomorphism relation

- For a complete metric space, compact = totally bounded (for each ϵ there is a finite cover consisting of open ϵ-balls).
- We can center the balls at points of the dense sequence; so being totally bounded is a Π^0_3 property of the atomic diagram.
- We say that a computable metric space is effectively compact if from a rational $\epsilon > 0$ we can compute how long an initial segment of the dense sequence is needed. E.g. $[0, 1]$.

Corollary

Isometry of compact metric spaces is a Π^0_3 relation on atomic diagrams.
Isometry of effectively compact metric spaces is a Π^0_2 relation on computable indices.

Complete Σ^0_n equivalence relations

We give natural examples of Σ^0_n equivalence relations that are complete for computable reducibility $\leq_c$. Results are due to Ianovski, Miller, Ng and N. (2012) unless mentioned otherwise.

n=1 The "precomplete" equivalence relation. For instance, equivalence of sentences under PA (Kripke and Pour-El, 1966).
n=2: Polynomial time Turing equivalence on exponential time sets.

n=3: recall $\cong_{comp}$ on computable Boolean algebras is complete for Σ^0_3 equivalence relations (Friedman, Fokina, and N). Another example: almost equality of r.e. sets.

n=4: Turing equivalence on r.e. sets.
for $n > 4$: jump equivalence $A^{(n-4)} \equiv_T B^{(n-4)}$ for r.e. sets A, B.

A Π^0_1 complete isomorphism relation

For a binary function f let xE_fy if $\forall u\, f(x,u) = f(y,u)$.

Theorem (Ianovski, Miller, Ng and N., 2012)

There is polynomial time computable function f such that each Π^0_1 equivalence relation is computably reducible to E_f.

Fix a finite alphabet $\mathbb{A}$ of size > 1. A predecessor tree is a nonempty subset of $\mathbb{A}^*$ closed under prefixes. Isomorphism of polynomial time predecessor trees is Π^0_1 by König's Lemma.
Given f from the theorem above, we code f_x into a polytime tree T_x: If $f_x(u) = k$ we add a leaf 1^u0^k to the tree. This yields:

Corollary

Isomorphism of polynomial time predecessor trees is complete for Π^0_1 equivalence relations.

There is no Π^0_2 complete equivalence relation

Theorem (Ianovski, Miller, Ng and N., 2012)

For $n \geq 2$ there is no Π^0_n and no Δ^0_n complete equivalence relation.

In particular, isometry of effectively compact metric spaces is not Π^0_2 complete.

To prove the theorem for $n = 2$: given a Π^0_2 equivalence relation E, we build a Δ^0_2 equivalence relation L with classes of size ≤ 2 such that $L \not\leq_c E$.

Problems. Classify the complexity of the following:

Isomorphism of computable Boolean algebras.

Goncharov and Knight (2002) proved this is Σ^1_1 -complete as a set of pairs.

Isomorphism of finitely presented groups.

Rabin proved that every equivalence class is Σ^0_1 complete as a set.
Conjecture: It is a Σ^0_1 complete equivalence relation.

Isomorphism of finite-automata presentable equivalence relations.

Kuske, Liu and Lohrey (2012) showed that this is Π^0_1 complete as a set of pairs.
Conjecture: It is a Π^0_1 complete equivalence relation.

Recent references

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- A. Melnikov. Computability and Structure. PhD thesis, Univ. of Auckland, 2012.
- These slides.

All available on my web page.