Demuth randomness and its variants

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Martin-Löf randomness is too weak

Martin-Löf (ML) randomness of a real is often considered too weak a randomness notion:

- ML-randomness allows for Turing completeness. But a random object shouldn't be able to compute the halting problem.
- ML-randomness of a real does not ensure that "almost everywhere" theorems of analysis hold at the real. For instance, it doesn't imply Lebesgue density 1 for effectively closed classes $\mathcal{P} \subseteq [0, 1]$ containing the real.

Lebesgue density

Let λ denote uniform (Lebesgue) measure.

Definition

Let $\mathcal{E}$ be a measurable subset of $[0,1]$. The (lower) density of $\mathcal{E}$ at a real z is

$$\rho(z \mid \mathcal{E}) = \liminf_{|J|\to 0} \frac{\lambda(J \cap \mathcal{E})}{|J|},$$

where J ranges over intervals with rational endpoints containing z.

This gauges how much of $\mathcal{E}$ is around z as we zoom in on z.

Theorem (Lebesgue Density Theorem, 1910)

Let $\mathcal{E} \subseteq [0,1]$ be measurable. For almost every $z \in [0,1]$:

if $z \in \mathcal{E}$, then $\mathcal{E}$ has lower density 1 at z.

Strengthening Martin-Löf

Strengthenings of Martin-Löf randomness have been proposed. For instance, a real z is 2-random if z is Martin-Löf random relative to the halting problem $\emptyset'$.

- A 2-random real z cannot compute $\emptyset'$ because z is random in $\emptyset'$.
- If $z \in \mathcal{P}$ where $\mathcal{P}$ is effectively closed, then $\mathcal{P}$ has density 1 at z.

But now we have overshot the goal:

- 2-randomness is already too strong to interact nicely with computability, or computable analysis.
- For instance, if z is 2-random, and $A \leq_{\mathrm{T}} z$ where A is Δ^0_2, then already A is computable. (That is, z doesn't compute interesting sets that are easily described.)

Finding the right randomness strength

We will discuss Demuth randomness and its variants.

Demuth $\Rightarrow$ weak Demuth $\Rightarrow$ balanced random $\Rightarrow$
difference random ($\Rightarrow$ ML-random)

- Even the weakest notion, difference randomness, implies Turing incompleteness.
- Balanced randomness implies density 1 for effectively closed classes.
- Even the strongest notion, Demuth randomness, is still not so strong to forbid interesting interaction with computability. For instance, there is a Δ^0_2 Demuth random.

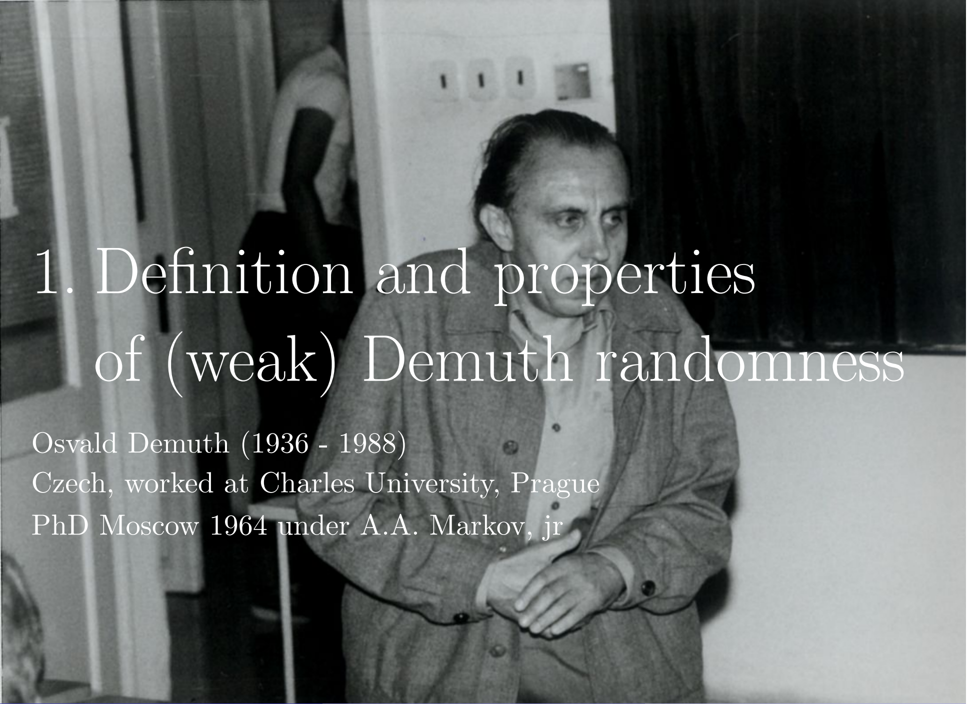
Main observations

These notions are useful for computable analysis.

Similar to 2-randomness we still obtain effective versions of “almost everywhere” theorems such as the Lebesgue density theorem for effectively closed classes.

Because they interact with computability, these notions can be used to solve problems on antirandomness / lowness properties:

in particular, they solve open problems on the K-trivial sets.



1. Definition and properties of (weak) Demuth randomness

Osvald Demuth (1936 - 1988)

Czech, worked at Charles University, Prague

PhD Moscow 1964 under A.A. Markov, jr

Definition from Demuth's 1982 paper[1]

Idea

- Demuth tests generalize Martin-Löf tests $(G_m)_{m\in\mathbb{N}}$: one can replace the m-th component (a Σ^0_1 class of measure $\leq 2^{-m}$) for a computably bounded number of times.
- A set $Z \subseteq \mathbb{N}$ fails a Demuth test if Z is in the final version of G_m for infinitely many m.

Demuth actually defined non-randomness.
He defined the WAP-sets, which stands for "weakly approximable in measure". He denoted this class of reals by $\mathcal{A}_\alpha$.

[1] O. Demuth. Some classes of arithmetical real numbers. Comment. Math. Univ. Carolin., 23(3):453 - 465, 1982

$$\overline{S}_0(q) \rightleftharpoons (S_0(q) \,\&\, \forall k\, (\mu_0(\mathcal{L}im\,(s_1^1(q, k+1))) \leq 2^{-k-1}))\,,$$

$$\mathcal{K}(p,q) \rightleftharpoons (\mathcal{K}_0(q) \,\&\, \forall k\, (!\langle p\rangle(k) \,\&\, \mathcal{M}is\,(s_1^1(q,k)) \leq \langle p\rangle(k)))\,,$$

где $\mathcal{K}$ одно из выражений S , $\hat{S}$ и $\overline{S}$,

б) если верно $S_0(q)$, то

$$\mathcal{V}_q \rightleftharpoons \bigcap_m \hat{\bigcup_{n \geq m}} [W_{\mathcal{L}im\,(s_1^1(q,n))}]\,,$$

$$\mathcal{V}_q^* \rightleftharpoons \hat{\bigcup_m} \bigcap_{n \geq m} [W_{\mathcal{L}im\,(s_1^1(q,n))}]\,,$$

$$\mathcal{Z}_q \rightleftharpoons \hat{\bigcup_k} [\mathcal{D}_{\mathcal{L}im\,(s_1^1(q,k))}]_c\,,$$

в) $${}^{p}\mathcal{A}_\alpha \rightleftharpoons \wedge X (\neg\neg\, \exists m\, (\hat{S}(p,m) \,\&\, X \in \mathcal{V}_m))\,;$$

г) $$\mathcal{A}_\alpha \rightleftharpoons \wedge X (\neg\neg\, \exists p q\, (\hat{S}(p,q) \,\&\, X \in \mathcal{V}_q))\,,$$

$$\mathcal{A}_\alpha^* \rightleftharpoons \wedge X (\neg\neg\, \exists p\, q\, (\hat{S}(p,q) \,\&\, X \in \mathcal{V}_q^*))\,,$$

$$\mathcal{A}_\beta \rightleftharpoons \mathcal{A} \setminus \mathcal{A}_\alpha\,.$$

(Weak) Demuth randomness

$[W_e]^{\prec}$ is the open set generated by W_e, viewed as subset of $2^{<\omega}$.

Definition

A Demuth test is a sequence of c.e. open sets $(S_m)_{m\in\mathbb{N}}$ such that

- $\forall m\, \lambda S_m \leq 2^{-m}$, and
- there is an ω-computably approximable (ω-c.a.) function f such that $S_m = [W_{f(m)}]^{\prec}$.

A real Z passes the test if $Z \notin S_m$ for almost every m. We say that Z is Demuth random if Z passes each Demuth test.

f ω-c.a. means $f(m) = \lim_t g(m,t)$ where g is computable and $g(m,\cdot)$ has only a computably bounded number of changes.
Let $S_m\langle t\rangle = [W_{g(m,t)}]^{\prec}$. This is the version of S_m at stage t.

We say that Z is weakly Demuth random if Z passes all Demuth tests $(S_m)_{m\in\mathbb{N}}$ with $S_m \supseteq S_{m+1}$. (Demuth's notation was $\mathcal{A}^*_\alpha$.)

Some notions between 2-randomness and ML-randomness

2-random

$\downarrow$

limit random = Schnorr random$[\emptyset']$

$\swarrow \quad \searrow$

weakly 2-random $\qquad$ Demuth random

$\searrow \quad \swarrow$

weakly Demuth random

$\downarrow$

Martin-Löf random

Existence and limitiations of Demuth randomness

For recent proofs see "Computability and Randomness", Sec. 3.6.

Proposition (Demuth '88)

There is a Δ^0_2 Demuth random set.

Let J^Z denote the Turing jump functional applied to oracle Z.

Theorem (Demuth '88, essentially)

Every Demuth random Z is GL_1 (i.e., $Z' \leq_T Z \oplus \emptyset'$).
In fact, there is a 2^x-computably approximable function f such that $J^Z(x)\downarrow$ implies $J^Z(x) \leq f(x)$.

That is, $f(x) = \lim_t g(x,t)$ with $\leq 2^x$ changes.

The proof doesn't work for weak Demuth randomness.

Not computably dominated

Recall that $A \subseteq \mathbb{N}$ is computably dominated (or, of hyperimmune-free degree) if each $g \leq_T A$ is dominated by a computable function.

Theorem (Miller and Nies; see Nies book, Section 8.1.)

A computably dominated set in GL_1 is not diagonally non-computable, and in particular not ML-random.

Corollary

No Demuth random set is computably dominated.

In contrast, there is a computably dominated weakly 2-random set.

A high Δ^0_2 weakly Demuth random

Recall that weak Demuth randomness means passing all Demuth tests such that $S_m \supseteq S_{m+1}$ for each m.

- Z is high if $\emptyset'' \leq_T Z'$, and superhigh if $\emptyset'' \leq_{tt} Z'$.
- A 2-random can be high (e.g., $\Omega^{\emptyset'}$ is high). But any Δ^0_2 Demuth random set Z is low ($Z' \leq_T \emptyset'$), and hence not high.

Kučera and Nies[2] studied computational complexity of weakly Demuth randoms.

Theorem (Kučera and Nies, 2011)

- There is a high Δ^0_2 weakly Demuth random set.
- No weakly Demuth random can be superhigh.

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[2] Demuth randomness and computational complexity. Ann. Pure Appl. Logic, 162:504 - 513, 2011.

Characterization by initial segments

N, Stephan, Terwijn 2005 proved that for $Z \subseteq \mathbb{N}$,

$$Z \text{ is 2-random} \iff \exists b \, \exists^\infty n \, [C(Z \upharpoonright_n) > n - b].$$

Theorem (Hölzl, Kräling, Stephan, Wu, 2010)

Z is Demuth random $\iff$

for each ω-c.a. function f, for almost every $b \in \mathbb{N}$:

$$\exists n \geq f(b) \; [C(A \upharpoonright_n) > n - b].$$

- They also prove that we may restrict ourselves to Demuth tests where all versions $S_m\langle t \rangle$ are effetively given clopen sets.
- The analog fails for monotone Demuth tests.

Lowness for randomness

- An oracle A is low for a randomness class $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^A$.
- That is, every $\mathcal{C}$ random is already $\mathcal{C}$-random relative to A.
- Characterizations are often hard and unexpected. For instance,

$$\text{low for Martin-Löf} = K\text{-trivial.}$$

Theorem (Downey and Ng, 2009)

If A is low for Demuth, then A is computably dominated.

Lowness for Demuth randomness

Theorem (Bienvenu, Downey, Greenberg, N, Turetsky)

Low for Demuth random = low for Demuth tests =
computably dominated & Demuth traceable.

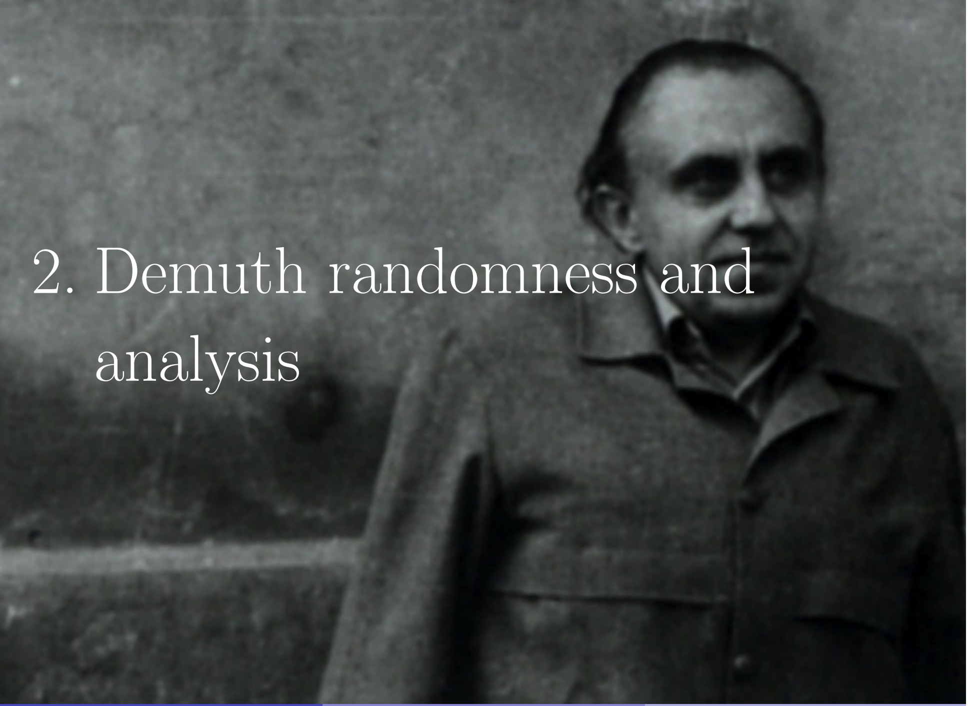
- A is Demuth traceable means: for any ω-c.a.-by-A function g, there is an ω-c.a. trace $(T_x)_{x \in \mathbb{N}}$, $|T_x| \leq 2^x$, with $g(x) \in T_x$.
- They built a perfect Π^0_1 class of Demuth traceable sets. Hence by Martin-Miller, low for Demuth has size continuum.

Corollary

low for Demuth $\subseteq$ computably dominated & jump traceable
$\subseteq$ low for Schnorr = computably traceable.

In the same paper, they showed that

$$\text{low for weak Demuth} = \text{computable}.$$



2. Demuth randomness and analysis

Demuth's original motivation

Let f be a real valued function with domain containing the computable reals in $[0, 1]$. We say that f is Markov computable if one can computably map any index for a computable Cauchy name of x to an index for a computable Cauchy name of $f(x)$.

- Demuth (1983) showed that for each Demuth random real $z \in [0, 1)$ the "Denjoy alternative" holds: either $f'(z)$ exists, or $+\infty = \limsup_{p<z<q,\, q-p\to 0}[f(q) - f(p)]/(p - q)$ and $-\infty = \liminf_{p<z<q,\, q-p\to 0}[f(q) - f(p)]/(p - q)$.
- He also showed that there is a Markov computable function f and a Martin-Löf-random real z such that the Denjoy alternative fails for f at z.

Difference randomness

- Difference randomness strengthens ML-randomness: the test components are differences of c.e. open sets.
- Franklin and Ng (2009): Let Y be ML-random. Then
$$Y \text{ is difference random} \iff Y \not\geq_T \Omega.$$
- Difference randomness can also be viewed as a further weakening of weak Demuth randomness: pass all monotonic Demuth tests (S_m) such that if $S_m\langle t_1\rangle \neq S_m\langle t_2\rangle$, then these versions of the m-th component are disjoint.

To show $\Omega \leq_{\mathrm{T}} Z \Rightarrow Z$ is not difference random:
Suppose $\Omega = \Gamma^Z$ for some Turing functional Γ. By a result of Miller-Yu, we may assume that for some $c \in \mathbb{N}$ at each stage t,

$$S_m[t] = \{Y : \Omega_t \restriction_{m+c} = \Gamma_t^Y \restriction_{m+c}\}$$

has measure at most 2^{-m}. Clearly Z fails this test.

Demuth randomness was way to strong

Theorem (Bienvenu, Hölzl, Miller, N, STACS 2012)

Let z be a difference random real. Then every Markov computable function f satisfies the Denjoy alternative at z.

- The classic Denjoy-Young-Saks theorem: for any $f\colon [0,1] \to \mathbb{R}$, the Denjoy alternative holds for almost every z.
- The standard proof (see e.g. Bogachev) uses porosity:

A set $\mathcal{P}$ is porous at a real z if $\exists \epsilon > 0$ $\exists$ arbitrarily small $\alpha > 0$ $(z - \alpha, z + \alpha)$ has subinterval of length $\geq \epsilon\alpha$ that is disjoint from $\mathcal{P}$.

- We showed as a crucial lemma: if $\mathcal{P}$ is effectively closed, and $z \in \mathcal{P}$ is difference random, then $\mathcal{P}$ is NOT porous at z.

Back to Lebesgue

- If $\mathcal{P}$ is porous at z via ϵ, then $\rho(z \mid \mathcal{P}) \leq 1 - \epsilon < 1$. So $\mathcal{P}$ does not have density 1 at z.
- We will now replace porosity by density.
- This seems like a digression from variants of Demuth randomness, but we'll get back to them soon.

Lebesgue Density Theorem

From HENRI LEBESGUE, Sur l' intégration des fonctions discontinues, Annales scientifiques de l' É.N.S. 3e série, tome 27 (1910), p. 361-450; p. 407.

> Raisonnant de même sur la densité à gauche, on voit finalement que *la densité d'un ensemble mesurable est égale à* un *en presque tous les points de cet ensemble.*

Translation:

Theorem (Lebesgue Density Theorem, 1910)

Let $E \subseteq [0,1]$ be measurable. For almost every $z \in [0,1]$:

if $z \in E$, then E has density 1 at z.

Turing incompleteness and positive density

Definition

We say that a real z is a positive density point if $\rho(z \mid \mathcal{P}) > 0$ for every effectively closed class $\mathcal{P} \ni z$.

The following result shows that to get positive density of a ML-random, its Turing incompleteness is exactly what we need.

Theorem (Bienvenu, Hölzl, Miller, N, STACS 2012)

Let z be a Martin-Löf random real. Then
z is NOT Turing above the halting problem $\iff$
z is a positive density point.

Using this to solve a long-standing open question

K denotes prefix free string complexity. K-trivial sets are far from random in a specific sense: there is b such that

$$\forall n\, K(A\restriction_n) \leq K(0^n) + b.$$

Many results assert that K-trivials are also close to computable.

Theorem (Day and Miller, 2011)

Let $A \subseteq \mathbb{N}$ be K-trivial. Suppose $Z \subseteq \mathbb{N}$ is a Martin-Löf random set such that $Z \oplus A \geq_T \emptyset'$. Then already $Z \geq_T \emptyset'$.

The idea is to translate Turing incompleteness of a ML-random random real into the downward closed property for Π^0_1 classes to have density 0 at the real.

A mystery notion: density-one points

Definition

We say that a real z is a density-one point if $\rho(z \mid \mathcal{P}) = 1$ for every effectively closed $\mathcal{P}$ containing z.

Question

Suppose z is Martin-Löf random. How much additional randomness is needed to ensure that z is a density-one point?

We will not answer this. However, we will see that it is closely connected to differentiability of effective nondecreasing functions.

Interval-r.e. functions

Definition

Let f be a non-decreasing function f on $[0, 1]$ with $f(0) = 0$. We say that f is interval-r.e. if $f(q) - f(p)$ is a left-r.e. real uniformly in rationals $p < q$.

If f is continuous, this implies lower semicomputable.
Recall that for $g\colon [0, 1] \to \mathbb{R}$ we let

$$V(g, [0, x]) = \sup \sum_{i=1}^{n-1} |g(t_{i+1}) - g(t_i)|,$$

where the sup is taken over all $t_1 \leq t_2 \leq \ldots \leq t_n$ in $[0, x]$.

Theorem (Freer, Kjos-Hanssen, N, Stephan, Rute 2012)

A continuous function f is interval-r.e. $\iff$
there is a computable function g such that $f(x) = \mathrm{Var}(g, [0, x])$.

Gauging the deviation from Turing completeness of a ML-random real z

z makes interval-r.e. functions differentiable	$\rightarrow$	z is a density one point	$\longrightarrow$	z is a positive density point
				$\updownarrow$
				$z \not\geq_T \emptyset'$

To see the first implication, suppose $\mathcal{P} \subseteq [0,1]$ is eff'ly closed.

- let $g(x) = \lambda([0,x) \cap \mathcal{P})$. Note that $\rho(z \mid \mathcal{P}) = \underline{D}g(z)$, and $1 - g$ is interval-r.e.
- If $z \in \mathcal{P}$ and $g'(z)$ exists, then $\rho(z \mid \mathcal{P}) = g'(z) = 1$, for otherwise z is not even ML-random.

Oberwolfach randomness

Definition

A monotonic Demuth test $(S_m)_{m\in\mathbb{N}}$ is called Oberwolfach test if $S_{m+1}\langle t\rangle$ changes twice in an interval of stages $\Rightarrow$ $S_m\langle t\rangle$ changes.

- OW randomness weakens balanced randomness of Figueira et al. (2011), where $S_n\langle t\rangle$ can change at most 2^n times.
- OW-random is just slightly stronger than ML-random.

For instance, recall that Y is LR-hard if every Y-random set is random relative to $\emptyset'$. Random Turing incomplete LR-hard sets exist, but are very close to Turing complete.

Theorem (Bienvenu et al. 2012)

If Y is ML-random but not OW-random, then Y is LR-hard.

In contrast, there is a low ML-random that is not balanced random (Figueira et al.), so balanced $\subset$ Oberwolfach.

OW-randomness and effective analysis

Theorem (Bienvenu, Greenberg, Kučera N, Turetsky 2012)

Let z be an Oberwolfach random real. Then

- every interval-r.e. nondecreasing function is differentiable at z.
- In particular, z is a density-one point.

No converses are known.

Randomness notions slightly stronger than ML

OW random	$\rightarrow$	z makes interval-r.e. functions differentiable	$\rightarrow$	z is a density one point	$\longrightarrow$	z is a positive density point
$\uparrow$						$\updownarrow$
$z \not\geq_{LR} \emptyset'$			$\rightarrow$			$z \not\geq_T \emptyset'$

For some of the randomness notions, the corresponding lowness notion is K-triviality. We know that Low(W2R,MLR) equals K-trivial. So they are all contained in K-trivial.

To prove that each K-trivial A is low for the notion, for the ML-random density one reals, adapt the existing proof of Day/Miller. For the ML-random LR-incomplete reals, use the fact (Nies book, 8.5.15) that if Z is ML-random, then $A \oplus Z \geq_{LR} \emptyset'$ implies $Z \geq_{LR} \emptyset'$.

The use of OW randomness

This is so far a hypothetical notion, which we haven't separated from difference randomness. But, it is useful all the same.

The covering problem (2005)

Is every K-trivial Turing below an incomplete ML-random?

We (Bienvenu et al., 2012) build a "smart" K-trivial A:

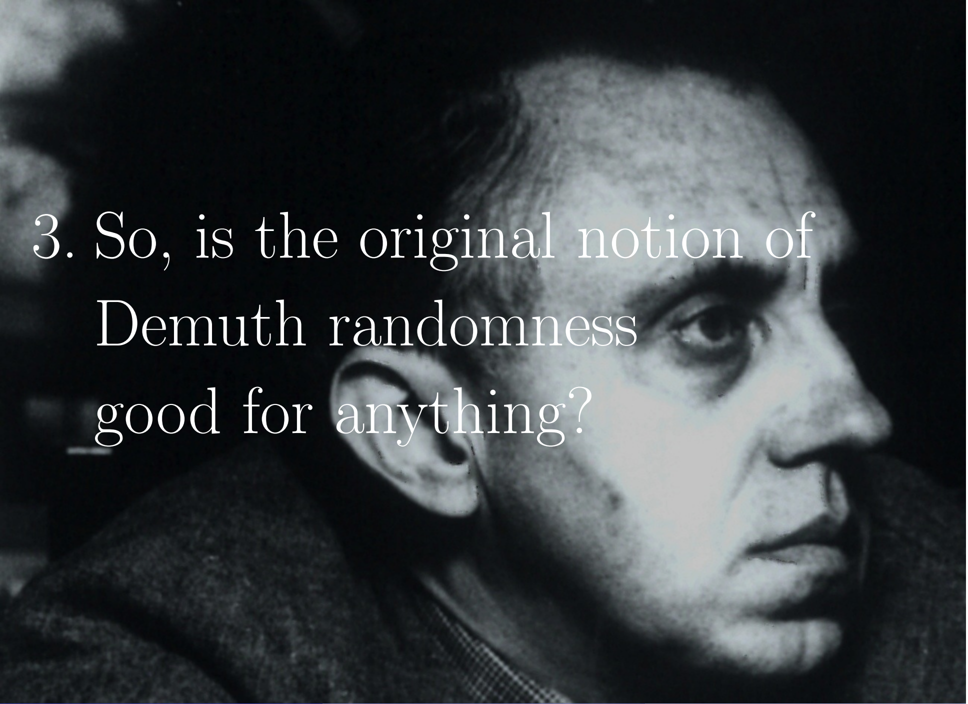
There is K-trivial A such that no $Y \geq_T A$ is Oberwolfach random.

On the other hand we show:

If B is K-trivial and Y is Martin-Löf but not Oberwolfach random, then $Y \geq_T B$.

Thus, if every K-trivial is below some incomplete ML-random Y, then we can fix this Y! The same Y does it for all.

3. So, is the original notion of Demuth randomness good for anything?



Using Demuth rdness to study extreme lowness

We say that A is strongly jump traceable (Figueira, N, Stephan 2005) if for each order function h, there is a c.e. trace (T_x) with $|T_x| \le h(x)$ such that $J^A(x)\downarrow$ implies $J^A(x) \in T_x$.

Theorem (Kučera and N, 2010)

If Y is Demuth random and $A \le_T Y$ for c.e. A, then A is strongly jump traceable.

In fact, for each order function h, they build a Demuth test $\mathcal{S}_h$ such that if Y passes $\mathcal{S}_h$, then each c.e. $A \le_T Y$ is h-jump traceable.
The converse:

Theorem (Greenberg and Turetsky, 2011)

If A is strongly jump traceable and c.e., then there is a Demuth random Y such that $A \le_T Y$.

Thus, for c.e. sets A, s.j.t $\Longleftrightarrow$ below a Demuth random.

Demuth test compatibility

Definition (N, 2011)

A property $\mathcal{E}$ of sets is called Demuth test compatible if every Demuth test is passed by some set Z with property $\mathcal{E}$.

Extending some methods of G,H,N (Adv. in Math, 2012):

Theorem (N, 2011)

Superlowness and superhighness are Demuth test compatible.

This yields a new proof of the result (GHN) that every set A below every superlow [superhigh] ML-random is strongly jump traceable: for each order function h, there is a superlow [superhigh] ML-random set Y passing the test $\mathcal{S}_h$ of Kučera/N (recall: Y passes $\mathcal{S}_h \Rightarrow$ each c.e. $A \leq_{\mathrm{T}} Y$ is h-jump traceable).

Bases for Demuth randomness

We say A is a base for Demuth randomness if $A \leq_{\mathrm{T}} Y$ for some Y that is Demuth random relative to A.

Theorem (N, 2011; Greenberg and Turetsky 2011)

Bases for Demuth randomness exist outside computable, and form a proper subclass of the strongly jump traceable. sets.

Summary

- Demuth randomness and its variants are strong enough to imply Turing incompleteness, yet not so strong to destroy interaction with computability
- The application Demuth had in mind can be achieved with the much weaker notion of difference randomness.
- Difference randomness and other notions still close to Martin-Löf yield further interaction with analysis (such as porosity and Lebesgue density), and thereby solve open problems on K-trivials.
- The original notion of Demuth randomness can be used to analyze extreme lowness properties.

References

- Demuth's path to randomness, with Kučera (2011)
- "Algorithmic aspects of Lipschitz functions" with Freer and Kjos-Hanssen, submitted.
- "The Denjoy alternative for computable functions", with Bienvenu, Hoelzl, and Miller, STACS 2012.
- "Characterizing strong jump traceability via randomness", with Greenberg and Hirschfeldt. Adv. in Math, to appear.
- these and other slides, on my web page.
- The logic blog available on my web site