The complexity of equivalence relations

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Sets versus equivalence relations

Consider a collection of mathematical objects. For instance, consider a class of countable, or of finite, structures for the same signature.

- The intuitive idea of a property of an object leads to sets.
- There are many ways to compare the complexity of sets; such as variants of many-one reducibility $\leq_m$.

- The intuitive idea of similiarity of two objects leads to equivalence relations.
- How should we compare their complexity?

Comparing complexity of equivalence relations

E equivalence relation on the collection of objects $\mathcal{X}$,
F equivalence relation on the collection of objects $\mathcal{Y}$.

Let

$$\boxed{E \leq_r F}$$

denote that there is an, in some sense effective, function $\phi\colon \mathcal{X} \to \mathcal{Y}$ such that

$$u\, E\, v \Leftrightarrow \phi(u)\, F\, \phi(v).$$

Thus ϕ induces an injection on equivalence classes $\mathcal{X}/E \to \mathcal{Y}/F$.

We will study this in the case that E, F are (possibly effective) isomorphism relations on structures in a countable language. Thus, we calibrate how hard it is to recognize isomorphism of structures in certain classes.

Countable structures, plain (classical) isomorphism

Countable structures, plain (classical) isomorphism

H. Friedman and Stanley, JSL 1989, introduced Borel reducibility $\leq_B$.

Question (Implicit in H. Friedman and Stanley, JSL 1989)

Which Borel (or even Σ^1_1) equivalence relations E are Borel reducible to the isomorphism relation on a class $\mathcal{K}$ of countable structures?

If so, we say that E admits a classification by countable structures.

Example: conjugacy of homeomorphisms of $[0, 1]$ mapping 0 to 0 admits a classification by countable structures (folklore).

Kechris and Louveau (1997) showed that E_1 (almost equality of sequences of real numbers) does not admit such a classification by countable structures.

Computable structures, plain isomorphism

Computable structures, plain isomorphism (1)

Fokina, S. Friedman, Harizanov, Knight, McCoy, Montalban 2010.

Consider a computable language. A structure with domain $\subseteq$ is called **computable** if its atomic diagram is computable. Let $I(\mathcal{K})$ be the set of recursive indices of computable structures in the class $\mathcal{K}$.

Note that isomorphism on $I(\mathcal{K})$ is Σ^1_1.
Their reductions between eqrels are given by a partial computable function with domain containing the relevant set $I(\mathcal{K})$. (This is sometimes denoted $\leq_{FF}$.)

Theorem (Fokina, S. Friedman et al., 2010)

Each Σ^1_1 equivalence relation is reducible in the sense of $\leq_{FF}$ to
(a) isomorphism on computable graphs
(b) isomorphism of computable subtrees of $\omega^{<\omega}$.

Computable structures, plain isomorphism (2)

Isomorphism for computable Boolean algebras is known to be

- m-complete for Σ^1_1 sets (Goncharov and Knight, 2002), but
- not known to be complete for Σ^1_1 eqrels.

$$
\begin{array}{ccccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \Sigma^0_3 & & \\
 & \subseteq & & \supseteq & & \subseteq & & \supseteq & & \subseteq & & \supseteq & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & & & \cdots \\
 & \supseteq & & \subseteq & & \supseteq & & \subseteq & & \supseteq & & \subseteq & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & \Pi^0_3 & &
\end{array}
$$

Figure: The arithmetical hierarchy of classes of subsets of $\mathbb{N}$.

- $\Delta^0_1 = \Sigma^0_0 = \Pi^0_0$ computable
- Σ^0_{n+1} means existential quantification of Π^0_n relations
- Π^0_{n+1} means universal quantification of Σ^0_n relations
- Δ^0_{n+1} means Σ^0_{n+1} and Π^0_{n+1}.

Automatic/finite structures, plain isomorphism

Automatic structures, plain isomorphism (1)

Theorem (Khoussainov, N, Rubin, Stephan, LICS 2004)

Isomorphism of automatic graphs is m-complete for Σ^1_1 sets.

This proof works by coding isomorphism of computable trees.

Now let's consider completeness for equivalence relations. By LICS 2004 and the 2010 result of FF + 4, we have

Theorem (after Khoussainov, N, Rubin, Stephan, LICS 2004)

Isomorphism of automatic graphs is complete for Σ^1_1 equivalence relations.

Automatic structures, plain isomorphism (2)

How about particular classes of automatic structures? E.g., isomorphism of automatic Boolean algebras is decidable (Khoussainov, N, Rubin, Stephan, LICS 2004).

Isomorphism is Π^0_1 for automatic equivalence relations, and for automatic trees of height 2 .

Theorem (Kuske, Liu, Lohrey, TAMS to appear)

For automatic equivalence relations/automatic trees of height 2, every Π^0_1 set can me computably reduced to the isomorphism relation.

N 2012 proved there is a complete Π^0_1 equivalence relation. It is not known whether any of these isomorphism relations is an complete equivalence relation.

Finite structures, plain isomorphism

Theorem (Boppana, J. Hastad and S. Zachos, 1987)

If isomorphism on finite graphs is NP complete as a set, then the polynomial time hierarchy collapses to Σ_2^p.

- The complexity of isomorphism on classes of finite structures was systematically studied by Buss, Chen, Flum, Friedman and Müller, JSL 2011.
- By the above, graph isomorphism is not polynomial time complete for NP equivalence relations unless PH collapses.

Computable structures, computable isomorphism

Computable structures, computable isomorphism

Theorem (S. Friedman, Fokina, N)

$\equiv_1$ *on r.e. sets is complete for* Σ^0_3 *equivalence relations.*

Corollary

Computable isomorphism on computable equivalence relations with all classes of size at most 2 is Σ^0_3 *complete.*

Proof of Corollary.

Given r.e. set A, build a computable equivalence relation R_A. Declare in advance that certain distinct elements t_i ($i \in \mathbb{N}$) are in different equivalence classes. If i enters A, attach a new element to t_i. □

Corollary

Computable isometry on computable metric spaces is Σ^0_3 *complete.*

Another example of Σ^0_3 completeness

Theorem (Fokina, Friedman, N)

Computable isomorphism of computable Boolean algebras is complete for Σ^0_3 eqrels.

Reduce 1-equivalence $\equiv_1$ of r.e. sets W^e containing the evens.

Define Boolean algebra B^e to be the interval algebra of a computable l.o. $\bigoplus_{x \in \omega} M^e_x$, where

- M^e_x has one element, until x enters W^e;
- when that happens, expand M^e_x to a computable copy of $[0, 1)_{\mathbb{Q}}$.

One can show that $W^e \equiv_1 W^i \Leftrightarrow B^e \cong_{comp} B^i$.

Π^0_2 equivalence relations

Theorem (Coskey, Hamkins, and R. Miller, ArXiv Feb. 2012)

Equality of r.e. sets is incomparable with the eqrel on r.e. sets of having the same median (middle element), or being both infinite, or both empty.

Consider the class $\mathcal{K}$ of permutations on $\mathbb{N}$ with all cycles finite. Then each computable structure in $\mathcal{K}$ is computably categorical, and (computable) isomorphism on $I(\mathcal{K})$ is Π^0_2.

Theorem (Egor Ianovski, 2012)

Isomorphism on $I(\mathcal{K})$ is computably equivalent to equality of r.e. sets.

A class of polynomial time structures with Π^0_1 complete isomorphism relation

For a binary function f let $xE_f y$ if $\forall u\, f(x,u) = f(y,u)$.

Theorem (Ianovski, N and Stephan, 2012)

There is polynomial time f such that each Π^0_1 eqrel is computably reducible to E_f.

Fix finite alphabet $\mathbb{A}$ of size > 1. A (predecessor) **tree** is a nonempty subset of $\mathbb{A}^*$ closed under prefixes. Isomorphism of polynomial time predecessor trees is Π^0_1 by König's Lemma.

Given f from the theorem above, we code f_x into a polytime tree T_x:
If $f_x(u) = k$ we add a leaf $1^u 0^k$ to the tree. This yields:

Corollary

Isomorphism of predecessor trees is complete for Π^0_1 eqrels.

And for recursion theorists...

Complete eqrels at various levels of the arithmetical hierarchy

I will now temporarily forget about structures, and discuss arithmetical equivalence relations from the point of view of a computability theorist.

$$
\begin{array}{ccccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \Sigma^0_3 & & \\
 & \subset & & \supset & & \subset & & \supset & & \subset & & \supset & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & & & \cdots \\
 & \subset & & \subset & & \subset & & \subset & & \subset & & \subset & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & \Pi^0_3 & &
\end{array}
$$

Figure: The arithmetical hierarchy.

Complete Σ^0_n equivalence relations

Clearly, for each $n \geq 1$ there is a complete Σ^0_n equivalence relation S. On the p-th column, let S be the transitive closure of the p-th Σ^0_n set of unordered pairs.

Natural examples of complete Σ^0_n equivalence relations?

$n = 1$ "precomplete" (or EUH) equivalence relation.

$n = 2$: Polynomial time Turing equivalence on exponential time sets is complete for Σ^0_2 eqrels (Ianovski and N).

$n = 3$: recall $\equiv_1$ on r.e. sets is complete for Σ^0_3 eqrels (F, F and N).

$n = 4$: Turing equivalence on r.e. sets is complete for Σ^0_4, and

No Π^0_2 complete...

Theorem (Miller and Ng)

For $n \geq 2$ there is no Π^0_n complete equivalence relation.

They only prove it for $n = 2$. By relativization proceed to higher n.

Theorem (S. Friedman, Fokina, and N 2012)

For each Σ^0_3 eqrel S there's a computable function g such that

$$\begin{aligned} ySz &\Rightarrow W_{g(y)} \equiv_1 W_{g(z)}, \text{ and} \\ \neg ySz &\Rightarrow W_{g(y)}, W_{g(z)} \text{ are Turing incomparable.} \end{aligned}$$

$\subseteq^*$ on r.e. sets is m complete for Σ^0_3 preorderings

For $X, Y \subseteq \omega$, we write $X \subseteq^* Y$ if $X \setminus Y$ is finite. We write $X =^* Y$ if $X \subseteq^* Y \subseteq^* X$.

Let W_e denote the e-th r.e. set.

Theorem

$\{\langle e, i\rangle : W_e \subseteq^* W_i\}$ *is m-complete for* Σ^0_3 *preorderings.*

Each eqrel is also a preordering. Thus, as an immediate consequence, we have

Corollary

$\{\langle e, i\rangle : W_e =^* W_i\}$ *is complete for* Σ^0_3 *eqrels*

Embeddability of subgroups of $(\mathbb{Q}, +)$

Note that for subgroups of $(\mathbb{Q}, +)$, bi-embeddability = isomorphism. So we can also strengthen a previous result.

Corollary

Computable embeddability among computable subgroups of $(\mathbb{Q}, +)$ *is m-complete for* Σ^0_3 *preorderings.*

Proof.

Let p_n be the n-th prime.
Code the e-th r.e. set W_e by the group generated by 1 and all $1/p_n$, $n \in W_e$. □

Some problems

Classify the complexity of the following:

- Completeness results for Δ^0_{n+1} equivalence relations?
- Isomorphism $G \cong H$ of finitely presented groups G, H.
  Conjecture: Σ^0_1 complete equivalence relation.
  Rabin proved that triviality (i.e., $G \cong \{e\}$) is Σ^0_1 complete as a set.
- Isomorphism and elementary equivalence of automatic structures (for the same signature).
  Conjecture: Π^0_1 complete equivalence relation.

Recent references

- Coskey, Hamkins, Miller: The hierarchy of equivalence relations on the natural numbers under computable reducibility. Computability, vol. 1, p. 15Ð38, 2012.
- Friedman, Fokina, Nies: Equivalence relations that are Sigma-3 complete for computable reducibility. Wollic 2012, to appear.
- Logic blog on my web page; these slides