Coding with first-order formulas

Talk at Universität Heidelberg on the occasion of Klaus Ambos Spies' 60th birthday

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Heidelberg 1989

In 1989, full of romantic ideas, I started my PhD under the supervision of Prof. Klaus Ambos-Spies in Heidelberg.

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This is where I worked.

The theory of $\mathcal{R}_{wtt}$

Klaus was interested in $\mathcal{R}_{wtt}$, the structure of recursively enumerable weak truth-table degrees.

Definition

$A \le_{\mathrm{wtt}} B$ if $A \le_{\mathrm{T}} B$ via a Turing reduction with a computable bound on the use. $\mathcal{R}_{wtt}$ is the partial order of wtt-degrees of r.e. sets.

The following was open:

Question

Is the first-order theory of $\mathcal{R}_{wtt}$ decidable?

Among the major reducibilities between $\le_m$ and $\le_T$, this was the only r.e. degree structure where the question remained open.

Undecidability

Mathematical Sciences Research Institute, Berkeley

Klaus participated in the logic year at MSRI 1989–1990. He invited me to come along for the latter part of it.

Two ingredients for the proof of $\mathrm{Th}(\mathcal{R}_{wtt})$ is undecidable

Theorem (Ambos-Spies and Soare, APAL 1989)

There is a uniformly c.e. sequence $(\mathbf{a}_i)_{i\in\mathbb{N}}$ of nonbounding Turing degrees so that any two of them form a minimal pair.

Richard Shore had the following result.

Theorem

Each Σ^0_3 ideal in $\mathcal{R}_{wtt}$ is of the form $[\mathbf{0},\mathbf{b}]\cap[\mathbf{0},\mathbf{c}]$ for some $\mathbf{a},\mathbf{b}\in\mathcal{R}_{wtt}$. (Call $\mathbf{b},\mathbf{c}$ and exact pair.)

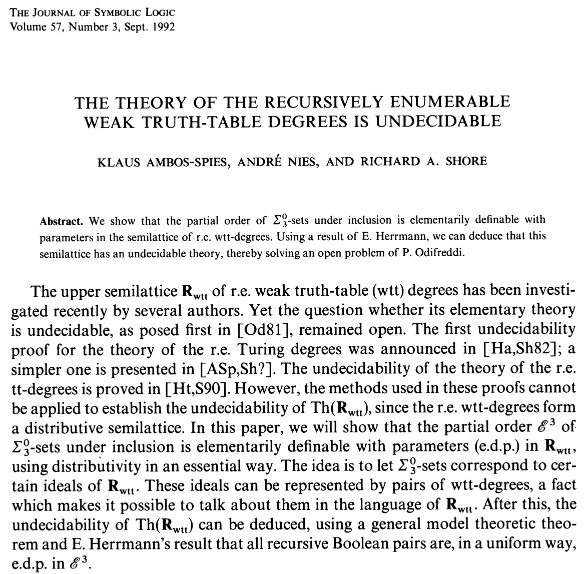
- We combine the second result with the wtt version of the first result. Pick an exact pair for the ideal generated by the $\mathbf{a}_i$.
- Then the set $\{\mathbf{a}_i : i\in\mathbb{N}\}$ is f.o. definable in $\mathcal{R}_{wtt}$ from parameters $\mathbf{b},\mathbf{c}$ as the maximal non-diamond degrees in the ideal. This uses that $\mathcal{R}_{wtt}$ is a distributive upper semilattice.

How to get undecidability

- Use further exact pairs to define any Σ^0_3 subset of $\{\mathbf{a}_i \colon i \in \mathbb{N}\}$.
- This establishes an interpretation with parameters $\mathbf{b}, \mathbf{c}$ in $\mathcal{R}_{wtt}$ of the lattice
$$\mathcal{E}^3 = (\Sigma^0_3, \subseteq).$$
- It was known by relativization of a result of Hermann (1984) that the theory of $\mathcal{E}^3$ is hereditarily undecidable: every non-empty subtheory is undecidable.
- This property of a theory is preserved under interpretations, even with parameters.

Thus, the theory of $\mathcal{R}_{wtt}$ is undecidable.

My first paper



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THE THEORY OF THE RECURSIVELY ENUMERABLE WEAK TRUTH-TABLE DEGREES IS UNDECIDABLE

KLAUS AMBOS-SPIES, ANDRÉ NIES, AND RICHARD A. SHORE

Abstract. We show that the partial order of Σ_3^0-sets under inclusion is elementarily definable with parameters in the semilattice of r.e. wtt-degrees. Using a result of E. Herrmann, we can deduce that this semilattice has an undecidable theory, thereby solving an open problem of P. Odifreddi.

The upper semilattice $\mathbf{R}_{\mathrm{wtt}}$ of r.e. weak truth-table (wtt) degrees has been investigated recently by several authors. Yet the question whether its elementary theory is undecidable, as posed first in [Od81], remained open. The first undecidability proof for the theory of the r.e. Turing degrees was announced in [Ha,Sh82]; a simpler one is presented in [ASp,Sh?]. The undecidability of the theory of the r.e. tt-degrees is proved in [Ht,S90]. However, the methods used in these proofs cannot be applied to establish the undecidability of $\mathrm{Th}(\mathbf{R}_{\mathrm{wtt}})$, since the r.e. wtt-degrees form a distributive semilattice. In this paper, we will show that the partial order $\mathscr{E}^3$ of Σ_3^0-sets under inclusion is elementarily definable with parameters (e.d.p.) in $\mathbf{R}_{\mathrm{wtt}}$, using distributivity in an essential way. The idea is to let Σ_3^0-sets correspond to certain ideals of $\mathbf{R}_{\mathrm{wtt}}$. These ideals can be represented by pairs of wtt-degrees, a fact which makes it possible to talk about them in the language of $\mathbf{R}_{\mathrm{wtt}}$. After this, the undecidability of $\mathrm{Th}(\mathbf{R}_{\mathrm{wtt}})$ can be deduced, using a general model theoretic theorem and E. Herrmann's result that all recursive Boolean pairs are, in a uniform way, e.d.p. in $\mathscr{E}^3$.

Drawbacks of this proof

- This undecidability proof for the theory of $\mathcal{R}_{wtt}$, while neat, relies on the hard fact that $\mathrm{Th}(\mathcal{E}^3)$ is hereditarily undecidable.
- Also, the quantifier complexity where the proof shows the theory of $\mathcal{R}_{wtt}$ to be undecidable is ridiculously high (about 12).

Quantifier level of undecidability

Undecidable fragments of $\mathrm{Th}(\mathcal{R}_{wtt})$

Theorem (Lempp, N, JSL 1995)

The $\forall\exists\forall\exists$ (4 quantifier) theory of $\mathcal{R}_{wtt}$ as a partial order is undecidable.

The proof still used the result of Ambos-Spies/Soare. We avoided $\mathcal{E}^3$ by giving a direct coding with first-order formulas of the finite bipartite graphs.

The ∀∃ theory is decidable

Theorem (Ambos-Spies, Fejer, Lempp, Lerman, JSL 1996)

The ∀∃ theory of $\mathcal{R}_{wtt}$ is decidable.

The gap of one quantifier alternation remains to this day.

Question

Is the ∀∃∀ (3 quantifier) theory of $\mathcal{R}_{wtt}$ as a partial order undecidable?

Also: do all non-trivial intervals of $\mathcal{R}_{wtt}$ have an undecidable theory?

The afterlife of a method (1)

The application to $\mathcal{R}_{wtt}$ of our method of interpreting $\mathcal{E}^3$ was superseded. Yet, modifications of this method have been very useful:

- We can replace the exact pairs by "exact degrees" that sort out subsets of the $\mathbf{a}_i$ of the right arithmetical complexity. This yields a uniform proof of undecidability for degree structures (N 1997).

The afterlife of a method (2)

- Suppose the ordering relation of the structure is Σ^0_k. We can replace the parameter definable set $\{\mathbf{a}_i : i \in \omega\}$ by a definable "effectively dense" Σ^0_k Boolean algebra $\mathcal{B}$; typically $\mathcal{B}$ is the splittings of some fixed element of the structure.

  We uniformly with parameters define the lattice of Σ^0_k ideals of $\mathcal{B}$ in the structure. This lattice has a hereditarily undecidable theory (N, 1997). So the theory of the structure is undecidable.

Applications to show undecidability of the theory:

- polynomial time m-degrees of time complexity classes properly including P; here k=2 (Downey, N 1996). This improved earlier result of Ambos-Spies and N that the theory of the poly time m-degrees of computable sets is undecidable.
- Intervals $[A, B]$ of r.e. sets $A \subset_m B$ (N, 97). More recently, Solovay degrees of left-r.e. reals (Downey, Griffiths, LaForte). Each time k=3.

How complex is the theory?

The complexity of the theory

- Once undecidability is known one asks what the actual complexity of the theory is.
- For arithmetical structures, an upper bound is the complexity of $\mathrm{Th}(\mathbb{N}, +\times)$. One wants to know whether this is the actual complexity.
- I showed this for $\mathcal{R}_m$, the r.e. many one degrees (1990), and with R. Shore (1992) for $\mathcal{R}_{tt}$, the r.e. truth table degrees.
- For $\mathcal{R}_{wtt}$, once again this took longest among the structures.

Theorem (N, APAL 2001)

The theory of $\mathcal{R}_{wtt}$ *has the same complexity as true first-order arithmetic* $\mathrm{Th}(\mathbb{N}, +\times)$.

How do such proofs work?

- The coding methods with parameters from the undecidability proof can be extended to a coding of a copy of $(\mathbb{N}, +\times)$.
  Given a list of parameters $\overline{\mathbf{p}}$, let $M_{\overline{\mathbf{p}}}$ denote the copy coded with this list.
  One now needs a first-order condition $\alpha(\overline{\mathbf{p}})$ expressing that $M_{\overline{\mathbf{p}}}$ is standard. This uses the context of the structure.
- There are various ways to do this:
  - Definability Lemmas for arithmetical sets. This uses that the standard part is arithmetical at a fixed level. R.e. many one degrees; lattice of r.e. sets.
  - Definable comparison maps to express that the model is smallest among all coded models. R.e. Turing degrees, $\mathcal{R}_{wtt}$.

Defining comparison maps between coded models in $\mathcal{R}_{wtt}$

Ambos-Spies (JSL 1984) proved an extension of the non-diamond theorem in the r.e. Turing degrees. He told me that this method shows:

Fact

Let $\mathbf{a} \in \mathcal{R}_{wtt}$ *be low. Then* $[\mathbf{a}, \mathbf{1}]$ *is non-complemented in* $\mathcal{R}_{wtt}$.

- Now consider arbitrary low degrees $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathcal{R}_{wtt}$ pairwise joining to $\mathbf{1}$.
- Then the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is definable, using as parameters an exact pair $\mathbf{b}, \mathbf{c}$ for the Σ^0_3 ideal $[\mathbf{0}, \mathbf{a}_1] \cap \ldots \cap [\mathbf{0}, \mathbf{a}_n]$.
- This easy definability of certain finite sets can be used to get definable comparison maps between coded copies of $(\mathbb{N}, +\times)$.

Some other distributive structures

Harrington and N (Adv in Math 1998) showed that $\mathrm{Th}(\mathcal{E})$, the theory of the lattice of r.e. sets, also interprets true arithmetic.

Recent results: Lewis, N and Sorbi (CiE 2009)/ Shafer 2009 showed independently that the Medvedev and Muchnik lattices have the highest possible complexity: the same as second-order analysis, i.e., second-order theory of $(\mathbb{R}, +\times)$. (Equivalently, third-order arithmetic.)
Shafer (APAL, to appear) showed the theory of the Medvedev lattice of Π^0_1 classes interprets true arithmetic, and the Muchnik lattice is undecidable.
Note that for Muchnik, the reducibility is merely Π^1_1.

Question

Determine the complexity of the lattice of Π^1_1 sets under inclusion.
Is it $\mathcal{O}^{(\omega)}$?
Do the same for the Muchnik lattice of Π^0_1 classes.

Automorphism bases of the r.e. degrees

Automorphism bases

Definition

A subset F of a structure is called automorphism base if every automorphism of the structure is determined by its action on F.

This is interesting also from the model theoretic point of view. For instance, consider a countable structure. Then

it has less than continuum many automorphisms $\Leftrightarrow$

it has a finite automorphism base.

Example

In the vector space $\mathbb{Q}^n$, every base is an automorphism base.

Automorphism bases in the r.e. Turing degrees

We still don't know whether the structure $\mathcal{R}_T$ of r.e. Turing degrees has a finite automorphism base.

A result of Ambos-Spies that fascinated me:

Theorem

Every nontrivial interval $[\mathbf{0}, \mathbf{c}]$ *is an automorphism base of* $\mathcal{R}_T$*.*

Thus, every r.e. degree is determined by its interaction with the arbitrarily small interval $[\mathbf{0}, \mathbf{c}]$.

For the proof Klaus Ambos-Spies introduced the "downward splitting property". Let $\hat{\mathbf{a}} = [\mathbf{0}, \mathbf{a}]$.

$$\mathrm{DSP} = \{\boldsymbol{a} : \forall \boldsymbol{b} \not\leq \boldsymbol{a} \ \forall \boldsymbol{d} \not\leq \boldsymbol{b} \vee \boldsymbol{a} [\widehat{\boldsymbol{d}} \cap \widehat{\boldsymbol{a}} \neq \widehat{\boldsymbol{d}} \cap \widehat{\boldsymbol{b}}]\}$$

This is discussed in my 2000 paper "Global Properties of Degree Structures", where my version of the proof is given.

Finite automorphism base of $\mathcal{R}_T$?

To obtain a finite automorphism base of $\mathcal{R}_T$, one can try to

- find a definable coding $M_{\overline{\mathbf{p}}}$ of copies of $\mathbb{N}$,
- and a definable map g from $M_{\overline{\mathbf{p}}}$ onto a nontrivial interval $[\mathbf{0}, \mathbf{c}]$.

Then the finitely many parameters needed would be an automorphism base of $\mathcal{R}_T$.
This is open despite years of trying (in the past).

Mapping onto end intervals

What we know is the dual:

Theorem (N, J. Math. Logic, 2003)

For every r.e. degree $\mathbf{d} \neq \mathbf{0}$*, there is a copy* $M_{\overline{\mathbf{p}}}$ *of* $\mathbb{N}$ *and a definable map* g *from* $M_{\overline{\mathbf{p}}}$ *onto* $[\mathbf{d}, \mathbf{1}]$.

The proof starts from the construction of "Slaman-Woodin sets" in the N-Shore-Slaman 1996 paper on definability in $\mathcal{R}_T$.

While this doesn't give a finite automorphism base, it still has interesting consequences:

- The restriction of an automorphism of $\mathcal{R}_T$ to $[\mathbf{d}, \mathbf{1}]$ is arithmetical at a fixed level.
- Each $\emptyset$-definable set generates a $\emptyset$-definable ideal. Example: the ideal generated by tops of diamonds is definable.
- Low_1 is parameter definable.

Randomness

The Ambos-Spies/Kučera open questions paper

- Klaus Ambos-Spies and Antonin Kučera had a paper entitled “Randomness in computability theory” in the 2000 Boulder Volume.
- The area developed so rapidly after this that in 2006 a new open questions paper was necessary! (Miller and N, BSL)

One important question in the 2000 paper was about oracle sets that are “low for random”. That is, they can’t derandomize reals when used as an oracle. The paper asked whether such sets must be below the halting problem.

I answered this in the affirmative. Later on (with Hirschfeldt) I showed that

low for random = K-trivial.

Randomness and degree theory

The new concepts yield interesting new examples, results and questions regarding complexity of sets and Turing degrees.

- Construction of a K-trivial is priority free solution to Post
- K-trivials form natural Σ^0_3 ideal in the structure $\mathcal{R}_T$ of r.e. degrees. (Question: does it have an exact pair in $\mathcal{R}_T$?)
- Extreme lowness properties such as strongly jump traceable can be defined via randomness, and are interesting degree-theoretically. For instance, if an r.e. set A is strongly jump traceable and Y is superlow ($Y' \leq_{\mathrm{tt}} \emptyset'$), then $A \oplus Y$ is superlow.

Maybe, the new notions shed light on the old problems, such as whether there is a definable map from a model $M_{\overline{\mathbf{p}}}$ onto an interval $[\mathbf{0}, \mathbf{c}]$.

References

- My 1998 Habilitation Thesis at Uni Heidelberg, "Coding Methods in Computability Theory and Complexity Theory". On my website, along with > 30 papers related to coding.
- Global properties of degree structures. Logic, Methodology and Philosophy of Science, P. Gardenfors e.a. (eds), Vol I, 65-80 (2000).
- These slides, on my web page.
- Books on Algorithmic Randomness (Downey/Hirschfeldt 2010; N 2009)