

Ten years of triviality

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The Incomputable, Chicheley Hall, 2012



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K -trivials: synopsis

During the last 10 years, the combination of algorithmic randomness and computability led to a novel class: the K -trivial sets.

- ▶ They are at the same time far from random, and close to computable.
- ▶ Coincidences with other classes.
- ▶ Construction of an incomputable c.e. K -trivial is injury-free.
- ▶ The K -trivials are closed downward under \leq_T .
- ▶ Σ_3^0 ideal in the (c.e.) Turing degrees, contained in superlow.

Several problems on the relationship of K -trivials to Martin-Löf randoms have been solved recently, or appear close to a solution. This relies on surprising new connections of Martin-Löf randomness with the analytic concept of density in a closed class.

Descriptive string complexity K

Consider a partial computable function from binary strings to binary strings (called machine). It is called **prefix-free** if its domain is an antichain under the prefix relation of strings.

There is a **universal** prefix-free machine \mathbb{U} :
for every prefix-free machine M ,

$$M(\sigma) = y \text{ implies } \mathbb{U}(\tau) = y \text{ for some } \tau \text{ with } |\tau| \leq |\sigma| + d_M,$$

and the constant d_M only depends on M .

The prefix-free Kolmogorov complexity of string y is the length of a shortest \mathbb{U} -description of y :

$$K(y) = \min\{|\sigma| : \mathbb{U}(\sigma) = y\}.$$

Definition of K -triviality

In the following, we identify a natural number n with the string that is its binary representation. For a string τ , up to additive const we have $K(|\tau|) \leq K(\tau)$, since we can compute $|\tau|$ from τ .

Definition (going back to Chaitin, 1975)

An infinite sequence of bits A is K -trivial if, for some $b \in \mathbb{N}$,

$$\forall n [K(A \upharpoonright_n) \leq K(n) + b],$$

namely, all its initial segments have minimal K -complexity.

It is not hard to see that $K(n) \leq 2 \log_2 n + O(1)$.

$$Z \text{ is random} \quad \Leftrightarrow \quad \forall n [K(Z \upharpoonright_n) > n - O(1)]$$

$$A \text{ is } K\text{-trivial} \quad \Leftrightarrow \quad \forall n [K(A \upharpoonright_n) \leq K(n) + O(1)]$$

Thus, being K -trivial means being far from random.

Why prefix-freeness?

Why not use plain Kolmogorov complexity C ?

- ▶ $\forall n [C(Z \upharpoonright_n) > n - O(1)]:$ there is no such Z (Katzev).
- ▶ $\forall n [C(A \upharpoonright_n) \leq C(n) + O(1)]:$ the only such A are the computable sets (Chaitin, 1976).

Early results on the K -trivials (before 2002)

- ▶ Chaitin (1975) proved that for each constant b there are only $O(2^b)$ K -trivials. This implies that each K -trivial set is Turing below the halting problem \emptyset' .
- ▶ Solovay (1976) built an incomputable K -trivial set.
- ▶ Zambella (1990 ILLC technical report), Kummer (unpubl), and Calude-Coles (1999) all gave examples of such sets that were computably enumerable.

The awakening of triviality (2002-2004)

- ▶ Downey, Hirschfeldt, Nies, and Stephan¹ gave a simple “cost function” construction of an incomputable c.e. K -trivial set. The construction was similar to the construction of a low for random set due to Kučera and Terwijn (1999).
- ▶ Downey et al. also showed that no K -trivial is Turing equivalent to the halting problem \emptyset' . For this they introduced what was later called the decanter method ².

¹Trivial reals. Proceedings of the 7th and 8th Asian Logic Conferences, Singapore University Press (2003), 103-131.

²Downey, R., Hirschfeldt, D., Nies, A and Terwijn, S. Calibrating randomness. Bull. Symb. Logic. 12 (2006), 411-491.

Building an incomputable c.e. K -trivial set A

Let $K_s(w) = \min\{|\sigma| : \mathbb{U}(\sigma) = w \text{ in } s \text{ steps}\}$.

To make A non-computable, we may have to put a number x of our choice into A at a stage s in order to diagonalize against the e -th computable function that claims to be the characteristic function of A .

In this case, for any $w \in \mathbb{N}$, $w > x$, we have to provide short descriptions, of length $K_s(w) + O(1)$, of the new $A_s \upharpoonright_w$.

For each e , eventually we can choose a number x so large that

$$\sum_{w=x+1}^s 2^{-K_s(w)} \leq 2^{-e-2}.$$

Because $\sum_e 2^{-e-2} = 1/2$, we will never run out of measure for new descriptions in the prefix-free machine we are building.

Lowness for Martin-Löf randomness

The following specifies a sense in which a set A is computationally weak when used as an oracle.

Definition

A is **low for Martin-Löf randomness** if every ML-random set Z is already ML-random with oracle A .

- ▶ This property was introduced by Zambella (1990).
- ▶ Kučera and Terwijn (1999) built a c.e. incomputable set of this kind.

Far from random = close to computable

Theorem (N, Adv. in Math., 2005)

Let $A \subseteq \mathbb{N}$. Then

A is K -trivial $\iff A$ is low for Martin-Löf randomness.

- ▶ The implication “ \implies ” is joint with Hirschfeldt.
- ▶ The proof combined the decanter with the “golden run” method, where an oracle \emptyset'' would be needed to find the node in a tree of runs that satisfies the conditions required.
- ▶ Since low for ML-randomness is closed downward under \leq_T , this implies that the K -trivials are closed downward under \leq_T .
- ▶ Nies also used the golden run method to show that each K -trivial A is superlow ($A' \leq_{tt} \emptyset'$), and truth-table below a c.e. K -trivial.

Bases for randomness (2005-2006)

Definition (Kučera, 1993)

A is a **base for randomness** if $A \leq_T Y$ for some Y that is Martin-Löf random relative to A .

Theorem (Hirschfeldt, N, Stephan, publ. 2007)

Base for randomness $\iff K$ -trivial.

Corollary

Let Y be an incomplete Martin-Löf random. Let $A \leq_T Y$ be a c.e. set. Then A is K -trivial.

Converse?

Covering question

Is every (c.e.) K -trivial below an incomplete ML-random?

The years 2007 -2010

- ▶ No progress on the covering question and related questions, such as non-cupping by ML-randoms.
- ▶ Results of Kučera and Slaman:
 - ▶ there is a low set Turing above all the K -trivials.
 - ▶ They used K -trivials to show that in every Scott set, every non-computable member is Turing incomparable with some other member.
- ▶ Bienvenu and Downey (STACS 2009),
Bienvenu, Merkle and N (STACS 2011): Solovay functions.

Solovay functions

Definition (going back to Solovay, 75)

A computable function $f: \mathbb{N} \rightarrow \mathbb{N}$ is a **Solovay function** if $\forall n K(n) \leq f(n)$, and infinitely often equal (all within constants).

Theorem (Bienvenu, Downey, Nies, Merkle)

Let f be any Solovay function. Then

$$A \text{ is } K\text{-trivial} \iff \forall n K(A \upharpoonright_n) \leq^+ f(n).$$

Program: redo theorems on K -trivials using Solovay functions. E.g, there is a new proof that every K -trivial is Turing below a c.e. K -trivial due to Bienvenu, 2010, unpubl.

The non-cupping question

Theorem (N, 2007; improved by N and Hirschfeldt)

There is an incomputable c.e. A such that

() $A \oplus Y \geq_T \emptyset'$ for ML-random Y implies $Y \geq_T \emptyset'$.*

Any such A is K -trivial.

Non-cupping question

Does the property (*) hold for every K -trivial set A ?

2010 - now

- ▶ Franklin and Ng (2010) show

difference random \iff ML-random & Turing incomplete.

- ▶ Bienvenu, Hölzl, Miller and N (STACS 2012) characterize difference randomness of a ML-random real via the analytic concept of **positive density** in an effectively closed class.
- ▶ Day and Miller (2011) solve the noncupping question. They use BHMN '12 for the implication " \implies " of their result:

K -trivial \iff Martin-Löf noncuppable.

- ▶ Bienvenu, Greenberg, Kučera, N and Turetsky (2012) build a "smart" K -trivial A :

every ML-random $Y \geq_T A$ is LR-hard (close to T-complete).

Positive density in a Π_1^0 class

Let $z \in \mathcal{P} \subseteq [0, 1]$, where \mathcal{P} is a Π_1^0 class. Let $\beta \in (0, 1]$.

We say that z has (lower) density β in \mathcal{P} if the portion of \mathcal{P} around z as we zoom in is eventually β :

$$\beta = \liminf_{p < z < q, q-p \rightarrow 0} \frac{\lambda[\mathcal{P} \cap (p, q)]}{q - p}.$$

Lebesgue's density theorem: a.e. $z \in \mathcal{P}$ has density 1 in \mathcal{P} .

Theorem (Bienvenu, Hölzl, Miller and N)

*Let z be ML-random. Then z is difference random \iff
 z has positive density in every Π_1^0 class $\mathcal{P} \ni z$.*

Day-Miller used this analytic understanding of Turing incompleteness among the ML-randoms for the non-cupping question.

Bienvenu, Greenberg, Kučera, N and Turetsky

We introduced “Oberwolfach randomness” of a real (2012), a notion possibly stronger than difference randomness. It implies having density 1 in Π_1^0 classes containing the real (full Lebesgue).

An effective null set (test) is given by a computable sequence of positive rationals $(\beta_x)_{x \in \mathbb{N}}$ with $\beta = \sup_x \beta_x$ finite, and a uniformly Σ_1^0 sequence $(V_x)_{x \in \mathbb{N}}$ with the conditions $V_x \supseteq V_{x+1}$ and $\lambda V_x \leq \beta - \beta_x$.

We build a “smart” K -trivial A :

There is K -trivial A such that no $Y \geq_T A$ is Oberwolfach random.

Recall that Y is **LR-hard** if every Y -random set is random relative to \emptyset' . Random incomplete LR-hard sets exist, but are very close to Turing complete. We improved a result of BHMN 2012:

If Y is ML-random but not OW-random, then Y is LR-hard.

More developments

Bienvenu, Greenberg, Kučera, Miller, N and Turetsky (2012):

If a Martin-Löf random z has lower density < 1 in some Π_1^0 class $\mathcal{P} \ni z$, then z computes every K -trivial set.

Barmalias and Downey (2012) showed that the ideal of K -trivials has no exact pair in the c.e. degrees.

Melnikov and N (Proc. AMS, in press) extended K -triviality to functions, and then to points in a computable metric space.

Recent references

- ▶ “The Denjoy alternative for computable functions”, with Bienvenu, Hoelzl, and Miller, STACS 2012. “ K -triviality, Oberwolfach randomness, and differentiability”. Oberwolfach preprint by BGKNT.



- ▶ My book “Computability and Randomness”. Updated and affordable paperback version has appeared (March 2012).
- ▶ Book by Downey and Hirschfeldt (2010).