Randomness via computability theory and effective descriptive set theory

André Nies

The University of Auckland

http://www.cs.auckland.ac.nz/nies

June 20, 2006

Understanding a set $Z \subseteq \mathbb{N}$

1. how much does Z know? (computational complexity). Either place Z in a level of an absolute complexity hierarchy, for instance recognizing that the set is computable, c.e., ..., Δ^1_1, Π^1_1.

Or measure its relative computational complexity, by comparing it to other sets, via $\leq_m$, $\leq_T$, ...

2. how well is Z organized? (degree of randomness). Again there is an absolute hierarchy, consisting of notions such as computably random, Martin-Löf-random, ..., Δ^1_1-ML random, Π^1_1-ML-random.

And there are ways to compare the degree of randomness of sets, using $\leq_S$, $\leq_K$, ...

$K(y)$

- A *machine* is a partial recursive function $M : \{0,1\}^* \mapsto \{0,1\}^*$.
- M is *prefix free* if its domain is an antichain under inclusion of strings.

Let $(M_d)_{d \geq 0}$ be an effective listing of all prefix free machines. The standard universal prefix free machine U is given by

$$U(0^d 1\sigma) = M_d(\sigma).$$

The prefix free version of Kolmogorov complexity is

$$K(y) = \min\{|\sigma| : U(\sigma) = y\}.$$

Thus, $K(y)$ is the length of a shortest prefix free description of y.

Initial segments

$$Z \upharpoonright n = Z(0) \ldots Z(n-1)$$

We often try to understand sets by looking at their finite initial segments.

- For computational complexity we have the use principle (a computation relative to Z only depends on finitely much of Z)
- for the degree of randomness we look at the length of descriptions of the finite initial segments.

Martin-Löf randomness

- Consider only null classes that are effectively c.e., namely, of the form $\bigcap_n R_n$, where
  - $R_n \subseteq 2^\omega$ is uniformly Σ^0_1 and
  - $\mu R_n \leq 2^{-n}$.
- Z is ML-random if Z is not in any effectively c.e. null class.

Schnorr's Theorem:

Z is Martin-Löf random iff for some c, $\forall n\ K(Z \upharpoonright n) \geq n - c$.

Example of a ML-random set:

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is a universal prefix free machine.

Connections

I discuss results connecting

- computability and randomness.
- effective descriptive set theory and randomness.

Direction "$\rightarrow$": Use concepts from the left side to restrict null classes, in order to obtain test concept that is effective in some sense.

Direction "$\leftarrow$": use concepts related to randomness to obtain interesting objects (examples): Ω, the class of K-trivials, ...

References

- September 2006 BSL: Survey "Calibrating randomness" (DHNT) and "Computability and randomness: open questions" (Miller/N)
- Survey "Eliminating concepts" in IMS Singapore volume: concentrates on K-trivials and lowness properties
- Upcoming books by Downey/Hirschfeldt and by N
- Hjorth/Nies, "Randomness via effective descriptive set theory", to appear in LMS

http://www.cs.auckland.ac.nz/nies

K-trivial sets

A set A is K-trivial if there is $c \in \mathbb{N}$ such that

$$\forall n\ K(A \upharpoonright n) \leq K(0^n) + c$$

(Chaitin, 1975).

- By Schnorr's theorem, Z is ML-random if for each n, $K(Z \upharpoonright n)$ is near its maximal value $n + K(0^n)$.
- To be K-trivial means to be far from ML-random, because $K(A \upharpoonright n)$ is minimal (all up to constants).

Solovay (1976) constructed a noncomputable K-trivial.

The cost function construction gives a noncomputable c.e. example.

Cost function construction

Downey, Hirschfeldt, Nies, Stephan 2001 gave a short "definition" of a noncomputable c.e. K-trivial set, which had been anticipated by various researchers (Kummer, Zambella). We use the "cost function"

$$\boxed{c(x,s) = \sum_{x<y\leq s} 2^{-K_s(y)}}.$$

This determines a non-computable set A:

$A_s = A_{s-1} \cup \{x : \exists e$

$W_{e,s} \cap A_{s-1} = \emptyset$	we haven't met e-th non-computability requirement
$x \in W_{e,s}$	we can meet it, via x
$x \geq 2e$	make A co-infinite
$c(x,s) \leq 2^{-(e+2)}\}$	ensure A is K-trivial.

Properties of the K-trivials

The K-trivial sets form an ideal $\mathcal{K}$ in the Δ^0_2 Turing degrees. It has the following properties (N, "Lowness properties and randomness", Adv. in Math 197, 2005):

- $\mathcal{K}$ is the downward closure of its c.e. members
- each $A \in \mathcal{K}$ is super-low: $A' \leq_{\mathrm{tt}} \emptyset'$.

A story that repeats

A lowness property of a set A specifies a sense in which the set is computationally weak.

The following has happened 4 times, so far.

- A research group G introduces a lowness property $\mathcal{L}$, and shows there is a non-computable c.e. set $A \in \mathcal{L}$.
- $\mathcal{L}$ turns out to be the same as $\mathcal{K}$.

Often the property says "Low for $\mathcal{C}$", where $\mathcal{C}$ is a reasonable class (for instance, rd'ness notion): A is low for $\mathcal{C}$ if $\mathcal{C}^A = \mathcal{C}$.

A as an oracle does not change $\mathcal{C}$.

1. Low for ML-random

For instance, Zambella introduced the class of low for MLrandom sets A (i.e., $\mathsf{MLRand}^A = \mathsf{MLRand}$)

- Kučera, Terwijn 1998 showed existence of a noncomputable c.e. low for ML-random
- N 2003 showed coincidence with low for K.

2. Low for K

Andrej A. Muchnik (1999) defined A to be low for K if

$$\forall y\ K(y) \leq K^A(y) + O(1),$$

and proved that there is a c.e. noncomputable A that is low for K.

- Low for K $\Rightarrow$ K-trivial is easily seen.
- Hirschfeldt and N, modifying the proof in (N 2003) that $\mathcal{K}$ is closed downwards, proved the converse direction.
- Low for K $\Rightarrow$ low for ML-random is also easy, using Schnorr's Theorem in relativized form.

3. Bases for ML-randomness

Kucera (APAL, 1993) studied sets A such that

$$A \leq_T Z \text{ for some } Z \in \mathsf{MLRand}^A.$$

That is, A can be computed from a set random relative to it.

We will call such a set a basis for ML-randomness. Each low for ML-random set A is a basis for ML-randomness.

- There is a noncomputable c.e. basis for ML-randomness (Kučera 1993).
- Each basis for ML-randomness is low for K (Hirschfeldt, N, Stephan, "Using random sets as oracles", ta).

Existence

Theorem[Kučera] *Let Z be Δ^0_2 and ML-random. Then there is a non-computable c.e. set $A \leq_T Z$.*

Apply this to a low ML-random set (say), and use that

Lemma 1 (Hirschfeldt.N.Stephan ta) *If $Z <_T \emptyset'$ is ML-random and A is c.e. and $A \leq_T Z$, then Z is already ML-random relative to A.*

4: low for weak 2-randomness

Z is weakly 2-random if Z is in no Π^0_2 null class.

Each such Z is ML-random.

A is low for weak 2-randomness if

weakly 2–random relative to A = weakly 2-random.

Theorem 2 (Downey,N,Weber,Yu 2005)

- *There is a c.e. noncomputable set that is low for weak 2-randomness*
- *each low for weak 2-randomness set is low for K.*

Also same as K-trivials!

Theorem 3 (N, Miller independently) *Each K-trivial set is low for weak 2-randomness.*

The two proofs are very different.

- Nies' proof used the golden run machinery that proved K-trivial = low for K
- Miller's proof is measure theoretic. He shows low for ML-randomness $\Rightarrow$ low for weak 2–randomness
- He used similar methods to show the coincidence of reducibilities: $\leq_{LR}$ is equivalent to $\leq_{LK}$ (a generalization of the fact that low for ML-random implies low for K). Here
  $A \leq_{LK} B$ if $\forall y\ K^B(y) \leq K^A(y) + O(1)$
  $A \leq_{LR} B$ if $\mathsf{MLRand}^B \subseteq \mathsf{MLRand}^A$.

ML-coverable

A set A is ML-coverable if there is a ML-random Z such that

$$A \leq_T Z <_T \emptyset'.$$

If A is also c.e., then Z is ML-random relative to A (H,N,S ta). Hence A is a basis for ML-randomness, so K-trivial.

Theorem (Kučera 1985) *Let Z be Δ^0_2 and ML-random. Then there is a noncomputable c.e. set $A \leq_T Z$.*

Taking Z low, we obtain a low noncomputable c.e. set A that is ML-coverable.

Question 4 *If a (c.e.) set A is K-trivial, is A ML-coverable?*

Almost complete sets

The following is dual to K-triviality.

B is almost complete if $\emptyset'$ is K-trivial relative to B. That is, there is $c \in \mathbb{N}$ such that

$$\forall n \; K^B(\emptyset' \upharpoonright n) \leq K^B(n) + c$$

Such a set is super-high: $\emptyset'' \leq_{\mathrm{tt}} B'$.

Existence

Theorem 5 (Jockusch/Shore 1983) *For each c.e. operator W there is a c.e. set C such that*

$$W^C \oplus C \equiv_T \emptyset'.$$

- Apply this to the c.e. operator W given by the cost function construction to obtain an incomplete but almost complete c.e. set.

Theorem 6 (N 2006) *The conclusion of the theorem also holds for "C ML-random".*

- Thus, there is a ML-random almost complete Δ^0_2-set.

The class $\mathcal{L}$

Let

$$\mathcal{L} = \{A : \ \forall Z \ (Z \text{ ML-random, almost complete } \Rightarrow \ A \leq_T Z\}.$$

- Hirschfeldt proved that there is a promptly simple set in $\mathcal{L}$.
- Since there is a ML-random almost complete Δ^0_2-set, each $A \in \mathcal{L}$ is ML-coverable, and hence K-trivial.
- Like $\mathcal{K}$, the class $\mathcal{L} \subseteq \mathcal{K}$ is an ideal.

Σ^0_3 null classes

The proof that there is a noncomputable c.e. set in $\mathcal{L}$ was simplified and generalized.

Theorem 7 (Hirschfeldt, Miller) *Let $\mathcal{C}$ be a Σ^0_3 null class. Then there is a noncomputable c.e. A such that $A \leq_T Z$ for each ML-random $Z \in \mathcal{C}$.*

Apply this to the Σ^0_3 null class $\mathcal{C}$ = almost complete in order to obtain a noncomputable c.e. $A \in \mathcal{L}$.

A proper subclass of the c.e. K-trivials?

- Figueira, N, Stephan (2004) introduced the following strengthening of super-lowness:
- For each function h that is computable, nondecreasing, unbounded, A' has an approximation that changes at most $h(x)$ times at x.
- They build a c.e. noncomputable such set, via a construction that resembles the cost function construction.
- Downey and Greenberg have announced that each c.e. set of this kind is K-trivial, and
- they form a proper subclass.

Effective descriptive set theory

Π^1_1 sets of numbers are a high-level analog of c.e. sets, where the steps of an effective enumeration are recursive ordinals. Hjorth and Nies (2005, Proc. LMS) have studied the analogs of K and of ML-randomness based on Π^1_1-sets.

- The Kraft-Chaitin theorem and Schnorr's Theorem still hold, but the proofs takes considerable extra effort because of limit stages
- There is a Π^1_1 set of numbers which is K-trivial (in this new sense) and not hyperarithmetic.

The classes are different now

Theorem 8 *If A is low for Π^1_1-ML-random, then A is hyperarithmetic.*

First we show that $\omega_1^A = \omega_1^{CK}$. This is used to prove that A is in fact K-trivial at some $\eta < \omega_1^{CK}$, namely

$$\forall n \; K_\eta(A \upharpoonright n) \leq K_\eta(n) + b.$$

Then A is hyperarithmetic, by the same argument Chaitin used in the c.e. case to show that K-trivial sets are Δ^0_2:

The collection of Z which are K-trivial at η form a hyperarithmetical tree of width $O(2^b)$ (because there are very few short descriptions). So Z is an isolated path.

Bases for ML-randomness

Consider the following reducibility $\leq_{\text{fin-h}}$, a possible high level analog of $\leq_T$:

- given by partial functions $\Phi : 2^{<\omega} \to 2^{<\omega}$ with Π^1_1 graph such that the domain is closed under prefixes, and, if $\Phi(t)\downarrow$, then $s \preceq t \Rightarrow \Phi(s) \preceq \Phi(t)$.
- $A \leq_{\text{fin-h}} Z$ if $\exists \Phi\ \forall n \exists m\ \Phi(Z \upharpoonright m) \succeq A \upharpoonright n$.

Kucera-Gacs: for each A there is ML-random Z such that $A \leq_{\text{fin-h}} Z$. Thus low for ML implies basis for ML (where "basis" is defined in terms of $\leq_{\text{fin-h}}$)

Theorem 9 *Each basis for ML-randomness is hyperarithmetical.*

A further randomness notion suggested by Martin-Löf

- In a little known paper (1970), Martin-Löf suggested the (lightface) Δ^1_1-classes of measure 0 as tests: Z is Δ^1_1-random if Z is in no null Δ^1_1-class.
- By an observation of Yu Liang, for each null Δ^1_1-class S one can find a Δ^1_1-ML-test $\{U_i\}_{i\in\mathbb{N}}$ such that $S \subseteq \bigcap_i U_i$.
- In particular, Π^1_1-ML-random implies Δ^1_1-random.
- Δ^1_1-random is the effective descriptive set theory analog of both computably random and Schnorr random.
- There is a Δ^1_1-random Z of slowly growing initial segment complexity (in sense of $K_{\Pi^1_1}$). Thus Z is not Π^1_1-ML-random.

A very strong rd'ness notion

Sacks (1990) in Exercise 2.5.IV suggested the Π^1_1 null classes as tests. This is the strongest randomness notion we have seen so far. The exercise was to separate this from Δ^1_1-random. While the exercise has been solved, only a few other things are known at present.

- Each Π^1_1-random Z satisfies $\omega_1^Z = \omega_1^{CK}$. In particular, the Π^1_1-ML-random set $\Omega_{\Pi^1_1}$ is not Π^1_1-random.
- By Gandy's basis theorem, some strongly random set satisfies $\mathcal{O}^Z \le_h \mathcal{O}$.
- Analog of van Lambalgen's Theorem

Theorem 10 (Hjorth,N) *There is a greatest Π^1_1-class $Q \subseteq 2^\omega$ of measure 0. Thus Q is a universal test for Π^1_1-randomness.*

Associated lowness notions

- Yu and Chong have announced that there is a perfect class of sets that are low for Δ^1_1-randomness. This contrasts with the Nies result that the only low for computably random sets are the computable ones.
- Not known if there is a nonhyperarithmetical set that is low for Π^1_1-randomness

Summary

- We have introduced the class of K-trivials (far from random) and the 4 classes: low for K, low for ML, basis for ML, low for weakly 2-random (all lowness properties, saying "computationally weak")
- All coincide
- In the effective descriptive set theory case, we have seen the concepts Δ^1_1-random $\supset$ Π^1_1-ML-random $\supset$ Π^1_1-random.
- K-trivial $\neq$ hyperarithmetic, while

  Low for Π^1_1-ML-random = base for Π^1_1-ML-random= hyperarithmetic