Recent results connecting computability and randomness

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Three questions

Progress on three open questions.

- (Reimann, Terwijn) Does every set of positive effective dimension compute a Martin-Löf-random set?
- (Stephan 2004) Is each K-trivial set Turing below an incomplete Martin-Löfrandom?
- (Kučera 2004) Is K-trivial the same as Martin-Löf non-cuppable?

All are in the open questions paper by Miller/N (to appear in BSL).

Part 1: Effective dimension

K denotes prefix free Kolmogorov complexity.

Schnorr's Theorem:

Z is Martin-Löf random iff for some c, $\forall n\ K(Z \upharpoonright n) \geq n - c$.

Example of a ML-random set:

$$\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|},$$

where U is a universal prefix free machine.

How about sets where $K(Z \upharpoonright n)$ is somewhat large? To quantify this, let

$$\underline{K}(Z) = \liminf_n K(Z \upharpoonright n)/n.$$

This is an effective version of the Hausdorff dimension of the class $\{Z\}$. If Z is ML-random then $\underline{K}(Z) = 1$, but the converse fails.

Examples

Here are two examples of sets Z of effective dimension $1/2$. They can be generalized to a rational $\alpha \in (0, 1)$.

- Let Y be ML-random, and let $Z = 0y_0 0y_1 0y_2 \ldots$.
- the Tadaki number

$$\Omega_{1/2} = \sum_{U(\sigma)\downarrow} 2^{-2|\sigma|}.$$

The question

Reimann and Terwijn (independently) observed that each of the known examples compute a ML-random set. So they asked the following:

Question 1 *If A has positive effective dimension, is there a ML-random set $Z \leq_T A$?*

This fails for wtt

Theorem 2 (with Reimann) *For each rational* $\alpha \in [0,1]$ *there is a set* $A \leq_{\mathrm{wtt}} \emptyset'$ *such that*

- $\underline{K}(A) = \alpha$ *and*
- $\underline{K}(Z) \leq \alpha$ *for each* $Z \leq_{\mathrm{wtt}} A$.

The result was also announced by Hirschfeldt and Miller.

If $\alpha < 1$, then no $Z \leq_{\mathrm{wtt}} A$ is ML-random.

- Let $\mathcal{P}$ be the Π^0_1-class given by

$$\mathcal{P} = \{Z : \forall n \geq n_0\, K(Z \upharpoonright n) \geq \lfloor \alpha n \rfloor\},$$

  where n_0 is chosen so that $\mu\mathcal{P} \geq 1/2$.

- Let $(\mathcal{P}_s)$ be an effective approximation by clopen classes, namely, $\mathcal{P} = \bigcap_s \mathcal{P}_s$.

We provide a lemma saying that the opponent has to invest a lot into the universal machine to completely remove a clopen class $\mathcal{C}$ from $\mathcal{P}$.

Lemma 3 *Let $\mathcal{C}$ be a clopen class such that $\mathcal{C} \subseteq P_s$ and $\mathcal{C} \cap P_t = \emptyset$ for stages $s < t$. Then $\Omega_t - \Omega_s \geq (\mu\mathcal{C})^\alpha$.*

For instance, if $\alpha = 1/2$, he has to spend $1/4$ to remove a class of measure $1/16$.

The setting

- Build A on P to ensure $\underline{K}(A) \geq \alpha$
- If Ψ_e is the e-th wtt reduction and $Z = \Psi_e^A$, then, for each rational $\beta > \alpha$, we have to satisfy

$$R_j : \exists r \geq j\, K(Z \upharpoonright r) \leq \beta r,$$

where $j > 0$ codes $\langle e, \beta \rangle$.

Strategy for R_j

Let $m_0 = 0$. R_j, $j > 0$, defines a number m_j and controls A in the interval $[m_{j-1}, m_j)$.

- First choose r_j large. Wait for the use bound $\psi_e(r_j)$ to converge and set m_j to be this use bound. (We may assume m_j is large enough, in particular $> m_{j-1}$.)

- The intent now is to define A in a way that we can compress $x = Z \upharpoonright r_j$. At stage s look for a large set $\mathcal{C} \subseteq \mathcal{P}_s$ of strings of length m_j all computing the same x. Compress x appropriately.

- If $\mathcal{C}$ falls off $\mathcal{P}$, then by the Lemma, the opponent has spent a lot. We can account our investment (to give a short description of x) against his, gaining the right to choose a new x.

Testing for totality

- x exist if Ψ_e^σ is defined for all relevant strings of length m_j. For in that case, if we partition this set of σ's into 2^{r_j} pieces depending on what Ψ_e^σ is, one piece must be large enough.
- In a first phase, before we even start to look for the right x, we try to choose $\sigma = A \upharpoonright m_j$ in a way that Ψ_e^A is partial. We need the fact that Ψ_e is wtt in order to make the number of changes in this first phase finite.

Part 2: Subclasses of the K-trivial sets

A set A is K-trivial if there is $c \in \mathbb{N}$ such that

$$\forall n \; K(A \upharpoonright n) \leq K(n) + c$$

(Chaitin, 1975).

- By Schnorr's theorem, Z is ML-random if for each n, $K(Z \upharpoonright n)$ is near its maximal value $n + K(n)$.
- To be K-trivial means to be far from ML-random, because $K(A \upharpoonright n)$ is minimal (all up to constants).

Cost function construction

Downey, Hirschfeldt, Nies, Stephan 2001 gave a short "definition" of a (promptly) simple K-trivial set, which had been anticipated by various researchers (Kummer, Zambella). We use the "cost function"

$$\boxed{c(x,s) = \sum_{x<y\leq s} 2^{-K_s(y)}}.$$

This determines a non-computable set A:

$A_s = A_{s-1} \cup \{x : \exists e$

$W_{e,s} \cap A_{s-1} = \emptyset$	we haven't met e-th simplicity requirement
$x \in W_{e,s}$	we can meet it, via x
$x \geq 2e$	make A co-infinite
$c(x,s) \leq 2^{-(e+2)}\}$	ensure A is K-trivial.

Properties of $\mathcal{K}$

The K-trivial sets form an ideal $\mathcal{K}$ in the Δ^0_2 Turing degrees. It has the following properties (N, "Lowness properties and randomness", 2003):

- $\mathcal{K}$ is the downward closure of its c.e. members
- $\mathcal{K}$ is Σ^0_3. Hence it is contained in $[\mathbf{o}, \boldsymbol{b}]$ for some c.e. low$_2$ $\boldsymbol{b}$: this is true for any Σ^0_3 ideal in the c.e. degrees (N ta, also see Downey/Hirschfeldt book)
- each $A \in \mathcal{K}$ is super-low: $A' \leq_{\mathrm{tt}} \emptyset'$.

A story that repeats

A lowness property of a set A says that A is computationally weak, in a particular sense.

The following has happened 3 times, so far.

- A research group introduces a lowness property $\mathcal{C}$, and shows there is a non-computable c.e. set $A \in \mathcal{C}$. (Usually, A is even promptly simple.)
- $\mathcal{C}$ turns out to be the same as $\mathcal{K}$.

1. Low for K

For instance, Andrej A. Muchnik (1999) defined A to be low for K if

$$\forall y \; K(y) \le K^A(y) + O(1),$$

and proved that there is a c.e. noncomputable A that is low for K.

- Low for $K \Rightarrow K$-trivial is easily seen.
- Hirschfeldt and N, modifying the proof in (N 2003) that $\mathcal{K}$ is closed downwards, proved the converse direction.

2. Low for ML-random

The same happened in the case of low for MLrandom sets A (i.e., $\mathsf{MLRand}^A = \mathsf{MLRand}$, a property implied by low for K.)

- Kučera, Terwijn 1998 showed existence
- N 2003 showed coincidence with low for K.

3. Bases for ML-randomness

Kucera (APAL, 1993) studied sets A such that

$$A \leq_T Z \text{ for some } Z \in \mathsf{MLRand}^A.$$

That is, A can be computed from a set random relative to it.

We will call such a set a basis for ML-randomness. Each low for ML-random set A is a basis for ML-randomness.

- There is a c.e. non-computable basis for ML-randomness (Kučera 1993).
- Each basis for ML-randomness is low for K (Hirschfeldt, N, Stephan, "Using random sets as oracles", ta).

Getting rid of K

Is there a characterization of K-trivial independent of randomness and K?

Figueira, N, Stephan have tried the following strengthening of super-lowness:

For each computable nondecreasing unbounded function h, A' has an approximation that changes at most $h(x)$ times at x.

They build a c.e. noncomputable such set, via a construction that resembles the cost function construction. No relationship to K-trivial is known.

Subclasses of $\mathcal{K}$

Does $\mathcal{K}$ have natural proper subclasses, or even subideals?

I will mostly restrict myself to the c.e. K-trivials. This is a minor restriction here, since there is a c.e. K-trivial set Turing above any K-trivial set.

One (not very convincing) example is

$$\mathcal{K} \cap \text{Cap},$$

the cappable K-trivials.

How about better examples? They should rather be defined by a single, natural property. I will discuss candidates. So far none of them are known to be proper subclasses.

Candidate 1: low for Π^0_2-random

Z is Π^0_2-random (or weakly 2-random) if Z is in no Π^0_2 null class. Each such Z is ML-random.

A is low for Π^0_2-random if

$$\Pi^0_2\text{-random relative to } A = \Pi^0_2\text{-random.}$$

Theorem 4 (Downey,N,Weber,Yu 2005)

- *There is a promptly simple low for Π^0_2-random set*
- *each low for Π^0_2-random set is low for K.*

Candidate 2: ML-coverable

A set A is ML-coverable if there is a ML-random Z such that

$$A \leq_T Z <_T \emptyset'.$$

If A is also c.e., then Z is ML-random relative to A (H,N,S ta). Hence A is a basis for ML-randomness, so K-trivial.

Theorem (Kučera1985) *Let Z be Δ^0_2 and ML-random. Then there is a promptly simple set $A \leq_T Z$.*

Taking Z low, we obtain a low promptly simple ML-coverable set A.

Question 5 (hard) *If a (c.e.) set A is K-trivial, is A ML-coverable?*

Candidate 3: ML-noncuppable

A Δ^0_2 set A is ML-cuppable if

$$A \oplus Z \equiv_T \emptyset' \text{ for some ML-random } Z <_T \emptyset'.$$

Many sets are ML-cuppable: If A is not K-trivial, then

- $A \not\leq_T \Omega^A$ (else A is a basis for ML-randomness, so K-trivial), and
- $A' \equiv_T \Omega^A \oplus A \geq_T \emptyset'$.

If A is also low, then $Z = \Omega^A <_T \emptyset'$, so A is ML-cuppable. This shows for instance that each c.e. non-K-trivial set A is ML-cuppable, since one can split it into low c.e. sets, $A = A_0 \cup A_1$, and one of them is also not K-trivial.

Existence

Theorem 6 (N, 2005) *There is a promptly simple set which is not ML-cuppable.*

The proof combines cost functions with the priority method. In fact I proved a stronger theorem, which implies the previous one by letting $Y = \Omega$.

Theorem 7 *Let $Y \in \Delta^0_2$ be Martin-Löf-random. Then there is a promptly simple set A such that, for each Martin-Löf-random set R,*

$$Y \leq_T A \oplus R \ \Rightarrow \ Y \leq_T R.$$

Barmparlias (ta) removed the hypothesis that Y be ML-random.

A common subclass

By recent work of Hirschfeldt and N, there is natural ideal $\mathcal{L}$ which is a subclass of both the ML-coverable and the ML-non-cuppable sets.

Key is the following notion. B is almost complete if $\emptyset'$ is K-trivial relative to B. That is, there is $c \in \mathbb{N}$ such that

$$\forall n \; K^B(\emptyset' \upharpoonright n) \leq K^B(n) + c$$

Such a set is super-high: $\emptyset'' \leq_{\mathrm{tt}} B'$.

I observed in the 2003 lowness properties paper that Jockusch-Shore inversion, applied to the c.e. operator W given by the cost function construction yields an incomplete but almost complete c.e. set.

Inverting a c.e. operator

Theorem 8 (Jockusch/Shore 1983) *For each c.e. operator W there is a c.e. set C such that*

$$W^C \oplus C \equiv_T \emptyset'.$$

Theorem 9 (N 2006) *The conclusion of the theorem also holds for "C ML-random".*

The proof combines

- methods of the Low Basis Theorem (to ensure $W^C \in \Delta^0_2$) with
- methods of the Kučera-Gacs theorem (to ensure $\emptyset' \leq_T W^C \oplus C$).

Corollary 10 *There is a ML-random almost complete Δ^0_2-set.*

The class $\mathcal{L}$

Let

$$\mathcal{L} = \{A : \forall Z\ (Z \text{ ML-random, almost complete } \Rightarrow\ A \leq_T Z\}.$$

- Hirschfeldt proved that there is a promptly simple set in $\mathcal{L}$.
- By the previous corollary, each $A \in \mathcal{L}$ is ML-coverable, and hence K-trivial.
- $\mathcal{L}$ is an ideal.

$\mathcal{L} \subseteq$ ML-noncuppable

The reason for this inclusion is that each potential ML-random cupping partner Z of a K-trivial A is almost complete:

$$\emptyset' \leq_T A \oplus Z,\ A \in \mathcal{K},\ Z \text{ ML-random} \Rightarrow Z \text{ almost complete.}$$

This takes a 4-line proof involving the van Lambalgen Theorem (Hirschfeldt 2005).

Comments on $\mathcal{L}$

Unless the unlikely coincidence $\mathcal{L} = \mathcal{K}$ holds, $\mathcal{L}$ is an ideal of the type we were looking for.

The existence proof for $\mathcal{L}$ was simplified and generalized.

Theorem 11 (Hirschfeldt, Miller) *Let $\mathcal{C}$ be a Σ^0_3 null class. Then there is a promptly simple A such that $A \leq_T Z$ for each ML-random $Z \in \mathcal{C}$.*

The proof is a simple cost function argument. (If $\mathcal{C}$ is the Π^0_2 class $\{Z\}$, for a ML-random Δ^0_2 set Z, then the proof turns into Kučera's proof.)

Apply this to the Σ^0_3 null class $\mathcal{C} =$ almost complete in order to obtain a promptly simple $A \in \mathcal{L}$.