Upper bounds for ideals in the Turing degrees

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The leading question

We study ideals in the c.e. Turing degrees.
The leading question is the following.

Let I be a proper ideal with a certain type of effective presentation.

What can we say about upper bounds of I in the c.e. degrees?

The ideal lattice of an usl U

- Let $(U, \leq \vee)$ be an uppersemilattice (usl).
- A set $I \subseteq U$ is an ideal if I is closed downwards an under the join operation $\vee$.
- An upper bound of an ideal I is a degree $\mathbf{b}$ such that $I \subseteq [\mathbf{0}, \mathbf{b}]$.

Some Facts:

- The set of ideals of U is a lattice, where the meet of I, J is the intersection, and the join of I, J is the ideal generated by $I \cup J$.
- An ideal I is called proper if $I \neq U$.
- Each $u \in U$ determines the ideal $\{x : \ x \leq u\}$, called a principal ideal.

Effective presentations of ideals

There are two interrelated approaches to effectively presenting an ideal I in the c.e. degrees.

(a) Require that I is generated by a uniformly c.e. sequence (possibly with further conditions).
We say that I is uniformly generated.

(b) Describe the index set $\Theta I = \{e\colon \text{ the degree of } W_e \text{ is in } I\}$ within the arithmetical hierarchy.
If ΘI is Σ^0_k etc. we say that I is a Σ^0_k ideal.

Basic Facts:

- The class of uniformly generated ideals is closed under join of ideals.
- Each principal ideal is Σ^0_4.
- For $k \geq 4$, the Σ^0_k ideals form a lattice.

Classes of ideals

For ideals, we have the implications

$$\Sigma^0_3 \Longrightarrow \text{uniformly generated} \Longrightarrow \Sigma^0_4.$$

It is not hard to show that the converse implications fail.

Definability and global properties

- Several natural ideals came out of the intrusion of randomness-related concepts into computability: K-trivial, strongly jump traceable, ...
- Earlier investigations of ideals focussed on their definability, and on the global properties of ideal lattices.
- A few proper ideals are known to be first-order definable without parameters in the c.e. degrees: the cappable degrees, and its subideal, the non-cuppable degrees.
- Nies (2001) showed that a definable set generates a definable ideal.
- Applying this, Yang Yue, Yu Liang, and Wang Wei found a few more examples of definable ideals: for instance, the ideal generated by the non-bounding degrees.

Back to the leading question

- By the Thickness Lemma every proper u.g. ideal has an incomplete upper bound.
- The Π^0_4 ideal of cappable degrees has no incomplete upper bound.
- What can we say about upper bounds of a proper Σ^0_3 ideal?
- How about a proper Σ^0_4 ideal?

Bounds for proper Σ^0_3 ideals

Theorem

Each proper Σ^0_3 ideal $\mathbb{I}$ in the c.e. degrees has a low$_2$ upper bound

- We first prove that each uniformly c.e. subsequence of a proper Σ^0_3 ideal is uniformly low$_2$.
- This uniform low$_2$-ness allows us to code all of $\mathbb{I}$ into an upper bound, while keeping this bound low$_2$.
- We have a $\emptyset''$ construction with a tree of strategies to read a low$_2$-ness index of the upper bound off the true path.

Corollary

There is a low$_2$ c.e. degree above all the K-trivials.

Bounds for proper Σ^0_4 ideals

Theorem

Each proper Σ^0_4 ideal $\mathbb{I}$ in the c.e. degrees has an incomplete upper bound.

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- The proof uses that there is a high c.e. set H of non-cuppable degree (Harrington and Miller 1981).
- We may assume that the degree of H is in $\mathbb{I}$.
- The construction now works because $\mathbb{I}$ is only $\Sigma^0_3(H)$.
- It is a $\emptyset''$ construction, but no explicit tree of strategies is needed. It suffices to use hat computations.

Prime ideals

The cappable degrees form a Π^0_4 prime ideal.
We can now answer a question of Calhoun (1990).

Corollary

No proper Σ^0_4 ideal is prime.

For, there is a minimal pair of degrees, none of which are below the upper bound of the ideal (Welch 1981).

Density of partial orders of ideals

Each principal ideal $[\mathbf{0}, \mathbf{b}]$, where $\mathbf{b} \neq \mathbf{0}$, has a maximal subideal that is $\Delta^0_4(\mathbf{b})$.
Choosing $\mathbf{b}$ low, this shows that the lattice of Σ^0_4 ideals is not dense.
In contrast, we have:

Theorem

The partial order of Σ^0_3 ideals in the c.e. degrees is dense.

In fact if $\mathbb{I}$ is a proper Σ^0_3 ideal in the c.e. degrees, then each degree $\mathbf{d} \notin \mathbb{I}$ splits in the quotient usl.

Some open questions on ideals

- Is every Σ^0_4 ideal $\mathbb{I}$ the intersection of the principal ideals it is contained in? (This would strengthen our result that $\mathbb{I}$ has an incomplete upper bound.)
- For $k \geq 4$, is the class of principal ideals definable in the lattice of Σ^0_k ideals?
- Let $\mathbf{K}$ be the ideal of K-trivial degrees. Are there c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{K} = [\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$?

Some references

- A. Nies, Parameter definable subsets of the recursively enumerable degrees, JML, 2002.
- Papers by Yang, Yu, Wang
- G. Barmpalias and A. Nies, Upper bounds on ideals in the Turing degrees, to appear.