Quasi-finitely axiomatizable groups

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December 5, 2004

Describing f.g. groups

We want to describe f.g. infinite groups by a finite amount of information. As many of them as possible.

If a finite, or even recursive, presentation exists, this does it.

I will discuss another way, based on first-order logic.

First–order logic

- The first–order language of groups consists of **formulas** built up from equations $t(x_1, \ldots, x_k) = 1$, using $\neg, \&, \vee, \rightarrow, \exists x, \forall x$.
- A **sentence** is a formula which has only bound variables.
- For a group G, $\mathrm{Th}(G)$ is the set of sentences which hold in G.

Examples (let [x,y] denote $x^{-1}y^{-1}xy$).

- The sentence $\forall x \forall y\ [x, y] = 1$ expresses that the group is abelian.
- The sentence

  $$\forall u, v \exists r, s, t\ [u, v] = r^2 s^2 t^2$$

  holds for all groups.

  (Let $r = u^{-1}v^{-1}$, $s = vuv^{-1}$, $t = v$).

G versus Th(G)

Th(G) contains the information whether G

- is nilpotent
- is torsion-free

But, many properties are not formalizable in first–order logic. For instance:

finitely generated, maximum condition (every subgroup is f.g.), simple...

Why first–order?

- A f.o. property of G is *intrinsic*. One does not have to go beyond G to verify it.
- Toolbox for first–order logic- compactness theorem, etc. There are fewer tools for more expressive languages.
- Th(G) is an interesting "invariant".

QFA

Another way to describe f.g. infinite groups by a finite amount of information:

G is said to be **quasi-finitely axiomatizable (QFA)** if there is a first order sentence φ such that G is, up to isomorphism, the only f.g. group satisfying φ.

Examples: Fix $m \geq 2$. Then the solvable Baumslag-Solitär group

$$\langle a, d \,|\, d^{-1}ad = a^m \rangle$$

is QFA. More on this later.

F.g. groups

Let G be finitely generated (f.g.) infinite. **To what extent is G determined by $\mathrm{Th}(G)$?**

By basic model theory:

- At the very least, one needs to require that G be countable. (There is some uncountable model of $\mathrm{Th}(G)$, since G is infinite.)
- There is in fact some other *countable* model. (Let n be the rank of G. Then $\mathrm{Th}(G)$ has infinitely many $n+1$–types and thus cannot be countably categorical.)

But, maybe, G is the only **f.g.** model of $\mathrm{Th}(G)$?

Quasi-axiomatizable

Definition. *An infinite f.g. group G is* **quasi-axiomatizable** *if, whenever H is a* f.g. *group with the same theory as G, then $G \cong H$.*

All f.g. abelian groups G are quasi-axiomatizable. For instance, if $G = \mathbb{Z}^n$, G is the only f.g. group such that

- G abelian, torsion free
- $|G/2G| = 2^n$.

These properties can be captured by an infinite axiom system.

Nilpotent groups

- Let $Z(G)$ denote the center of G
- class-1 nilpotent $\Leftrightarrow$ abelian
- G is nilpotent of class $c+1 \Leftrightarrow$ $G/Z(G)$ is nilpotent of class c.

QA for nilpotent groups

Theorem 1 (Hirshon (1977) and Oger (1990))

- *Each f.g. torsion-free class-2 nilpotent group is quasi-axiomatizable.*
- *There are f.g. torsion-free class-3 nilpotent groups* G, H *such that* $\mathrm{Th}(G) = \mathrm{Th}(H)$, *but* $G \not\cong H$.

Oger's part was: for f.g. nilpotent G, H,

$$\mathrm{Th}(G) = \mathrm{Th}(H) \;\Leftrightarrow\; G \oplus \mathbb{Z} \cong H \oplus \mathbb{Z}.$$

The direction $\Leftarrow$ is actually true for any groups G, H.

Hirshon (1977) had asked for which groups A can $\mathbb{Z}$ be cancelled from a direct product $A \oplus \mathbb{Z}$:

$$A \oplus \mathbb{Z} \cong B \oplus \mathbb{Z} \;\Rightarrow\; A \cong B.$$

- True for any group A which is f.g. torsion free **class-2** nilpotent,
- not always when A is f.g. torsion free **class-3** nilpotent.

QFA groups

We consider f.g. groups G where a single axiom, along with the information that G is f.g., suffices.

Definition 2 *An infinite f.g. group G is quasi-finitely axiomatizable (QFA) if there is a first–order sentence φ such that*

- $G \models \varphi$
- *If H is a f.g. group such that $H \models \varphi$, then $G \cong H$.*

Thus the axiom provides a characterization of G among the f.g. groups, using only elementary properties of G.

I will look at this property in various classes of groups: abelian, nilpotent, metabelian, and particular type of permutation groups.

Abelian groups are not QFA

- By quantifier elimination for the theory of abelian groups, each φ which holds in an abelian group G also holds in $G \oplus \mathbb{Z}_p$, for almost all primes p.
- If G is f.g. then $G \not\cong G \oplus \mathbb{Z}_p$, so G is not QFA.

Thus one *needs* an infinite axiom system to describe G.

An algebraic characterization

Sabbagh and Oger gave a beautiful algebraic characterization of QFA for infinite nilpotent groups G. Informally, G is QFA iff G is far from abelian. Let

- $G' = \langle [x, y] : x, y \in G \rangle$
- $\Delta(G) = \{x : \exists m > 0 \; x^m \in G'\}$
  (least N such that G/N is torsion free abelian).

Theorem 3 (Sabbagh and Oger, 2004)
G is QFA $\Leftrightarrow Z(G) \subseteq \Delta(G)$.

The direction $\Rightarrow$ holds for all f.g. groups.

For abelian G, $Z(G) = G$ and $\Delta(G)$ is the (finite) torsion subgroup. So $Z(G) \not\subseteq \Delta(G)$ (we assume G is infinite).

Heisenberg group

$$\mathrm{UT}_3^3(\mathbb{Z}) = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

is torsion free, $Z(G) = G' = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $\mathrm{UT}_3^3(\mathbb{Z})$ is QFA by Oger/Sabbagh. I had proved this first via a proof using an interpretation of $(\mathbb{N}, +, \cdot)$ due to Mal'cev.

All non-abelian UT groups over $\mathbb{Z}$, and all free nilpotent non-abelian groups are QFA.

$Z(G) \subseteq G'$ is not enough

Fix $m \geq 2$. Let $G = \begin{pmatrix} 1 & m\mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$

$Z(G)$ as before, but $G' = \begin{pmatrix} 1 & 0 & m\mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

G' doesn't include $Z(G)$, but $\Delta(G) = Z(G)$. The group is QFA.

Oger has shown that nilpotent & QFA is closed under subgroups of finite index. Also direct products.

Metabelian QFA groups

For groups G, A, C one writes $G = A \rtimes C$ if

$$AC = G,\ A \lhd G, \text{ and } A \cap C = \{1\}.$$

Give examples of QFA groups that are split extensions $A \rtimes C$, A abelian, $C = \langle d \rangle$ infinite cyclic.

Theorem 4 (N)

- *For each $m \geq 2$, the group*

  $$H_m = \langle a, d \,|\, d^{-1}ad = a^m \rangle$$

  is QFA (one-relator group)
- *For each prime p, the restricted wreath product $\mathbb{Z}_p \wr \mathbb{Z}$ is QFA (not finitely presented)*

Francis Oger has further examples of this type. This time A is f.g.

Structure of those groups

- H_m is a split extension of $A = \mathbb{Z}[1/m] = \{zm^{-i} : z \in \mathbb{Z}, i \in \mathbb{N}\}$ by $\langle d \rangle$, where the action of d is $u \mapsto um$.
- By the definition, $\mathbb{Z}_p \wr \mathbb{Z}$ is a split extension $A \rtimes C$, where
  - $A = \bigoplus_{r \in \mathbb{Z}} \mathbb{Z}_p^{(r)}$, $\mathbb{Z}_p^{(r)}$ is a copy of $\mathbb{Z}_p$
  - $C = \langle d \rangle$ with d of infinite order
  - d acts on A by shifting

QFA proofs

The proofs that those groups are QFA follow the same scheme. The group A is given by a first-order definition. One writes a list $\psi(d) = P1 \ \& \ \ldots \& Pk$ of first-order properties of an element d in a group G so that the axiom $\exists d \, \psi(d)$ shows that the group in question is QFA.

Let C be the centralizer of d, namely $C = \{x : [x, d] = 1\}$. In the following, u, v denote elements of A and x, y elements of C.

(P1) The commutators form a subgroup (so G' is definable)

(P2) A and C are abelian, and $G = A \rtimes C$

(P3) $C - \{1\}$ acts on $A - \{1\}$ without fix points. That is, $[u, x] \neq 1$ for each $u \in A - \{1\}$, $c \in C - \{1\}$.

(P4) $|C : C^2| = 2$

- To specify H_m one uses the definition $A = \{g : g^{m-1} \in G'\}$, and requires in addition that

(P5) $\forall u\, d^{-1}ud = u^m$,

(P6) The map $u \mapsto u^q$ is 1-1, where q is a fixed prime not dividing m

(P7) $x^{-1}ux \neq u^{-1}$ for $u \neq 1$

(P8) $|A : A^q| = q$

- To specify $\mathbb{Z}_p \wr \mathbb{Z}$, one uses the definition $A = \{g : g^p = 1\}$, and requires in addition that $|A : G'| = p$ and no element in $C - \{1\}$ has order $< p$.

QFA higher up

A set $S \subseteq \omega$ is called an **arithmetical singleton** if there exists a formula $\varphi(X)$ in the language of arithmetic extended by a new unary predicate symbol X that for each $P \subseteq \mathbb{N}$, $\varphi(P)$ is true in the standard model of arithmetic if and only if $P = S$.

Examples: all arithmetical sets; $\mathrm{Th}(\mathbb{N}, +, \cdot)$

Theorem 5 (Morozov, Nies) *For each arithmetical singleton $S \subseteq 3\mathbb{N}$, there exists a 3–generated QFA–group G_S whose word problem $W(G)$ has the same complexity as S (namely, $S =_T W(G)$).*

G_S is a subgroup of the permutations of $\mathbb{Z}$ generated by successor, $(0, 1)$ and

$$\prod_{k \in S}(k, k+1, k+2).$$

Prime groups

A notion from model theory:

- G is **prime** if G is an elementary submodel of each H such that $\mathrm{Th}(G) = \mathrm{Th}(H)$.
- For f.g. groups, this is equivalent to: there is a generating tuple $\overline{g}$ whose orbit (under the automorphisms of G) is definable by a first order formula.
- Uniquely determined by theory

Various theories fail to have one:
$\mathrm{Th}(\mathbb{Z}, +, 0), \mathrm{Th}(F_2)$.

Prime vs QFA

Oger and Sabbagh showed that for nilpotent f.g. G:

$$G \text{ is QFA } \Leftrightarrow G \text{ is prime.}$$

Theorem 6 (N) *There are uncountably many non-isomorphic f.g. groups that are prime. (The class consists of the f.g. groups satisfying a sentence* α*.)*

In particular, not all prime are QFA. The permutation groups G_S as above provide those examples (for sets S whose elements are sufficiently far apart).

Question 7 *Is each QFA group prime?*

Theory of classes of groups

Let $\mathcal{E}$ be a class of groups. The theory $\mathrm{Th}(\mathcal{E})$ *is the set of first–order sentences which hold for all members of $\mathcal{E}$.*

If $\mathcal{C} \subseteq \mathcal{D}$, then $\mathrm{Th}(\mathcal{C}) \supseteq \mathrm{Th}(\mathcal{D})$. One can gauge the expressiveness of first–order logic in group theory by asking:

for which proper inclusions $\mathcal{C} \subset \mathcal{D}$ of natural classes are the theories different?

QFA Criterion. *If there is a QFA group G in $\mathcal{D} - \mathcal{C}$ then* $\mathrm{Th}(\mathcal{C}) \supset \mathrm{Th}(\mathcal{D})$.

This is because the negation of the axiom for G is in $\mathrm{Th}(\mathcal{C}) - \mathrm{Th}(\mathcal{D})$.

As an example, consider $\mathcal{C}$ = "finite", $\mathcal{D}$= "f.p. with solvable WP". Let $G = \mathbb{Z}[1/2] \rtimes \mathbb{Z}$.